Recursive Estimation of Room Impulse Responses with Energy Conservation Constraints

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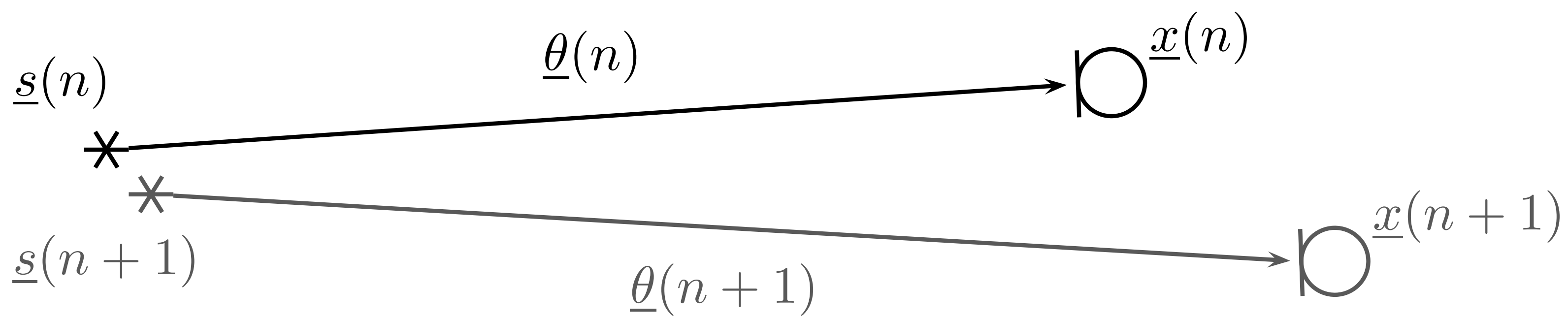
Introduction and Motivation (I)

- Goal: Track time-varying room impulse response (RIR) between source and microphone



- Signal model:

$$\underline{x}(n) = \underline{\theta}(n) * \underline{s}(n) + \underline{z}(n), \qquad n = 1, \ldots, N$$

with

$\underline{x}(n) \in \mathbb{R}^K$	...	measured microphone signal
$\underline{\theta}(n) \in \mathbb{R}^M$	...	room impulse response
$\underline{s}(n) \in \mathbb{R}^S$	...	known source signal
$\underline{z}(n) \in \mathbb{R}^K$	...	Gaussian noise, $\underline{z}(n) \sim \mathcal{N}(\underline{0}, \mathbf{C}(n))$, temporally uncorr.

Introduction and Motivation (II)

- Signal model (in Matrix notation):

$$\underline{x}(n) = \mathbf{\Theta}(n)\underline{s}(n) + \underline{z}(n) = \mathbf{S}(n)\underline{\theta}(n) + \underline{z}(n), \qquad n = 1, \dots, N$$

where $\mathbf{\Theta}(n) \in \mathbb{R}^{K\times S}$ and $\mathbf{S}(n) \in \mathbb{R}^{K\times M}$ are Toeplitz matrices with

$$\mathbf{\Theta}(n) = \begin{bmatrix} \underline{\theta}(n) & & & \mathbf{0} \\ & \underline{\theta}(n) & & \\ & & \ddots & \\ \mathbf{0} & & & \underline{\theta}(n) \end{bmatrix} \qquad \mathbf{S}(n) = \begin{bmatrix} \underline{s}(n) & & & \mathbf{0} \\ & \underline{s}(n) & & \\ & & \ddots & \\ \mathbf{0} & & & \underline{s}(n) \end{bmatrix}$$

- Popular solution approach: Weighted least squares (WLS) estimator

$$\hat{\underline{\theta}}_{\mathsf{WLS}}(n) = \arg\min_{\underline{\theta}} \sum_{i=0}^{n} \beta^{n-i} \left(\underline{x}(i) - \mathbf{S}(i)\underline{\theta}\right)^T \mathbf{C}(i)^{-1} \left(\underline{x}(i) - \mathbf{S}(i)\underline{\theta}\right)$$

where $0 \leq \beta \leq 1$ is the exponential forgetting factor

Introduction and Motivation (III)

- Idea: Improve estimation performance by incorporating a priori information

- Open Questions:
  - **What a priori information ($=$ constraint) can we exploit?**
    - Energy conservation constraint $\|\mathbf{\Theta}(n)\underline{s}(n)\|^2 \leq \|\underline{s}(n)\|^2$
  - **How to efficiently include the constraint into tracking algorithm?**
    - Recursive constrained maximum likelihood (RCML) estimator
    - Recursive affine minimax (RAMX) estimator
    - Recursive minimum mean squared error (RMMSE) estimator

Contents

Energy Conservation Constraint (I)

- Expressing the energy conservation mathematically, we need to ensure that

$$\|\mathbf{\Theta}(n)\underline{s}(n)\|^2 = \underline{s}(n)^T\mathbf{\Theta}(n)^T\mathbf{\Theta}(n)\underline{s}(n) \ \leq \ \underline{s}(n)^T\underline{s}(n) = \|\underline{s}(n)\|^2$$

  i.e.

  "signal energy at microphone" $\leq$ "signal energy at source"

  which implies

$$\mathbf{\Theta}(n)^T\mathbf{\Theta}(n) - \mathbf{I} \preceq \mathbf{0}$$

- Condition has to hold for all signal lengths S
  - if condition fulfilled for particular S_0 then also fulfilled for all $S < S_0$
  - therefore, it is sufficient to consider case $S \to \infty$

Energy Conservation Constraint (II)

- Two equivalent representations of the set $\Theta = \{\underline{\theta} : \mathbf{\Theta}(n)^T \mathbf{\Theta}(n) - \mathbf{I} \preceq \mathbf{0}\}$:
  - LMI representation: Using Schur's lemma, the constraint is

$$\begin{bmatrix} \mathbf{I} & \mathbf{\Theta}(n)^T \\ \mathbf{\Theta}(n) & \mathbf{I} \end{bmatrix} \succeq \mathbf{0}$$

    i.e. Θ is a convex set.

  - Frequency domain representation:
    - Use equivalence of the eigenvalues of $\mathbf{\Theta}(n)^T \mathbf{\Theta}(n)$ (bandlimited Toeplitz matrix) and corresponding circulant matrix for $S \to \infty$
    - Constraint can be written as

$$|\mathit{\Theta}(\omega, n)|^2 = \left| \sum_{m=0}^{M-1} \theta_m(n) e^{-j\omega m} \right|^2 \leq 1 \; \forall \, \omega \in [0, 2\pi)$$

      where $\mathit{\Theta}(\omega, n)$ is the room frequency response

Energy Conservation Constraint (III)

- All three recursive estimators are based on frequency domain representation

- Approximation of the frequency domain representation:

  - Use DFT to evaluate $\Theta(\omega, n)$ at discrete frequencies ω_l

$$\omega_l = \frac{2\pi l}{L} \quad \text{with} \quad l = 0, \ldots, L-1$$

  where $L \geq M$.

  - Let $\mathbf{P} \in \mathbb{R}^{M \times L}$ be matrix consisting of the first M rows of $\mathbf{I}$. Then,

$$|\Theta(\omega_l, n)|^2 = \underline{\theta}(n)^T \mathbf{P} \underline{\tilde{f}}_l \underline{\tilde{f}}_l^H \mathbf{P}^T \underline{\theta}(n) = \underline{\tilde{\theta}}^T \underline{\tilde{f}}_l \underline{\tilde{f}}_l^H \underline{\tilde{\theta}} \leq 1 \quad \forall\, l = 0, \ldots, \lfloor L/2 \rfloor$$

  where $\underline{\tilde{f}}_l \in \mathbb{C}^L$ is the lth column of the DFT matrix $\mathbf{U} \in \mathbb{C}^{L \times L}$, i.e.

$$\underline{\tilde{f}}_l^H = \begin{bmatrix} 1 & e^{-j\omega_l} & \ldots & e^{-j\omega_l (L-1)} \end{bmatrix}$$

Constrained Recursive Estimation (I)

- Consider the time-variant signal model

$$\underline{x}(n) = \mathbf{S}(n)\underline{\theta}_0(n) + \underline{z}(n), \quad n \geq 0$$

where it is assumed that $\underline{\theta}_0(n) \in \Theta \subset \mathbb{R}^M$

- Motivated by the WLS estimator, we reformulate this signal model into

$$\underline{x}_n = \mathbf{S}_n\underline{\theta}(n) + \underline{z}_n, \quad \underline{\theta}(n) \in \Theta$$

where $\underline{x}_n$, $\mathbf{S}_n$ and $\underline{z}_n$ are stacked versions of $\underline{x}(i)$, $\mathbf{S}(i)$ and $\underline{z}(i)$, i.e.

$$\begin{aligned}
\underline{x}_n &= \begin{bmatrix}\underline{x}(n)^T & \cdots & \underline{x}(0)^T\end{bmatrix}^T \\
\mathbf{S}_n &= \begin{bmatrix}\mathbf{S}(n)^T & \cdots & \mathbf{S}(0)^T\end{bmatrix}^T, \\
\underline{z}_n &= \begin{bmatrix}\underline{z}(n)^T & \beta^{-1/2}\underline{z}(n-1)^T & \cdots & \beta^{-n/2}\underline{z}(0)^T\end{bmatrix}^T.
\end{aligned}$$

- Noise $\underline{z}(n)$ Gaussian and temporally uncorrelated, i.e. $\underline{z}_n \sim \mathcal{N}(\underline{0}, \mathbf{C}_n)$ with

$$\mathbf{C}_n = \operatorname{diag}\left(\mathbf{C}(n), \beta^{-1}\mathbf{C}(n-1), \cdots, \beta^{-n}\mathbf{C}(0)\right)$$

Constrained Recursive Estimation (II)

- How can we avoid estimators with growing computational complexity?
  $\Rightarrow$ use concept of sufficient statistics

- A sufficient statistic for our signal model is given by

$$\underline{t}(\underline{x}_n) = \left(\mathbf{S}_n^T \mathbf{C}_n^{-1} \mathbf{S}_n\right)^{-1} \mathbf{S}_n^T \mathbf{C}_n^{-1} \underline{x}_n = \underline{\hat{\theta}}_{\mathsf{WLS}}(n)$$

- Efficiently computed by recursive weighted least squares algorithm (RWLS)

  - in each time step, it updates

$$\begin{aligned} \underline{\hat{\theta}}_{\mathsf{WLS}}(n) &= \underline{t}(\underline{x}_n) & \ldots \quad & \textit{sufficient statistic} \\ \mathbf{R}_n^{-1} &= \left(\mathbf{S}_n^T \mathbf{C}_n^{-1} \mathbf{S}_n\right)^{-1} & \ldots \quad & \textit{inverse correlaton matrix} \end{aligned}$$

$\Rightarrow$ both quantities are needed by the three recursive estimators
$\Rightarrow$ estimators are based on RWLS and use constraint to improve performance

Recursive Constrained ML (RCML)

- The RCML is given by

$$\underline{\hat{\theta}}_{\text{RCML}}(n) = \arg\max_{\underline{\theta}} \; p(\underline{x}_n|\underline{\theta}) \quad \text{s.t.} \quad \underline{\theta} \in \Theta$$

$$= \arg\min_{\underline{\theta}} \; (\underline{\theta} - \underline{\hat{\theta}}_{\text{WLS}}(n))^T \mathbf{R}_n (\underline{\theta} - \underline{\hat{\theta}}_{\text{WLS}}(n)) \quad \text{s.t.} \quad \underline{\theta} \in \Theta$$

At each time step, we have to check $\underline{\hat{\theta}}_{\text{WLS}}(n) \in \Theta$:

if yes: $\underline{\hat{\theta}}_{\text{RCML}}(n) = \underline{\hat{\theta}}_{\text{WLS}}(n)$

if no: find minimum of $(\underline{\theta} - \underline{\hat{\theta}}_{\text{WLS}}(n))^T \mathbf{R}_n (\underline{\theta} - \underline{\hat{\theta}}_{\text{WLS}}(n))$ on boundary of Θ

- For our RIR Tracking problem:
  - Constraint is given by

$$|\Theta(\omega_l, n)|^2 = \underline{\theta}(n)^T \mathbf{P} \underline{\tilde{f}}_l \underline{\tilde{f}}_l^H \mathbf{P}^T \underline{\theta}(n) \leq 1 \quad \forall\, l = 0, \ldots, \lfloor L/2 \rfloor$$

$\Rightarrow$ Quadratically constrained quadratic program (QCQP)
$\Rightarrow$ can be efficiently solved

Recursive Affine Minimax (RAMX)

- The RAMX is given by

$$\underline{\hat{\theta}}_{\text{RAMX}}(n) = (\mathbf{I} + \mathbf{M}(n))\,\underline{\hat{\theta}}_{\text{WLS}}(n) + \underline{u}(n)$$

where $\mathbf{M}(n)$ and $\underline{u}(n)$ are the solution to the minimax problem

$$\min_{\mathbf{M},\underline{u}} \max_{\underline{\theta}\in\Theta} \quad \|\mathbf{M}\underline{\theta} + \underline{u}\|^2 + \operatorname{tr}\left\{(\mathbf{I}+\mathbf{M})\mathbf{R}_n^{-1}(\mathbf{I}+\mathbf{M})^T\right\}$$

and the inverse correlation matrix $\mathbf{R}_n^{-1} = (\mathbf{S}_n^T\mathbf{C}_n^{-1}\mathbf{S}_n)^{-1}$ is updated by RWLS

- For our RIR Tracking problem:
  - reduce computational complexity by using $\underline{u}(n) = \underline{0}$ and $\mathbf{M}(n) = \alpha\mathbf{I}$
  - use S-procedure on Θ to reformulate optimization problem into SDP

Recursive Minimum Mean Squared Error

- RMMSE recasts frequentist problem into Bayesian one:
  - use uniform prior on Θ
  - can be motivated by "maximum entropy principle"
- Using Bayesian sufficient statistics for our signal model, RMMSE is given by

$$\underline{\hat{\theta}}_{\text{RMMSE}}(n) = \frac{\int_{\Theta} \underline{\theta} \exp\{-\frac{1}{2}(\underline{\theta} - \underline{\hat{\theta}}_{\text{WLS}}(n))^T \mathbf{R}_n^{-1}(\underline{\theta} - \underline{\hat{\theta}}_{\text{WLS}}(n))\} d\underline{\theta}}{\int_{\Theta} \exp\{-\frac{1}{2}(\underline{\theta} - \underline{\hat{\theta}}_{\text{WLS}}(n))^T \mathbf{R}_n^{-1}(\underline{\theta} - \underline{\hat{\theta}}_{\text{WLS}}(n))\} d\underline{\theta}}$$

- Integrals are solved using Monte Carlo integration:
  - Sampling from a posterior density using rejection sampling
  - Posterior density is truncated Gaussian density (support given by Θ)
- For our RIR Tracking problem:
  - use rejection sampling with $|\mathit{\Theta}(\omega_l, n)|^2 \leq 1$

Simulation Results (I)

- Task: Estimate RIR from moving source to fixed microphone

- Setup:
  - Source
    - Moves along a straight line, 10cm increment between two pos.
    - Overall 11 positions, i.e. source moves one meter in total
    - Source signal: Gaussian noise of length $S = 100$ samples
  - Room:
    - Room size: $3 \times 3 \times 2.3$m, $T_{60} = 120$ms
    - Image source model ($f_s = 12$kHz) results in RIRs with $M = 1241$ taps
  - Estimators:
    - For all three estimators: DFT length is $L = 2^{14} = 16\,384$
    - $I = 3\,000$ samples for RMMSE (MC approx. of integrals)

Simulation Results (II)

- Definitions:
  - Signal-to-noise ratio: SNR $= 10\log_{10}\frac{\|\mathbf{\Theta}(n)\underline{s}(n)\|^2}{(M+S-1)\sigma^2}$
  - normalized error measure: $E = \frac{1}{N}\sum_{n=1}^{N}\frac{\|\hat{\underline{\theta}}(n)-\underline{\theta}(n)\|^2}{\|\underline{\theta}(n)\|^2}$
- Instantaneous estimators:

SNR	WLS	CML	AMX	MMSE
5 dB	1.23$\times 10^{0}$	1.20$\times 10^{0}$	1.21$\times 10^{0}$	1.02$\times 10^{0}$
10 dB	3.88$\times 10^{-1}$	3.88$\times 10^{-1}$	3.87$\times 10^{-1}$	3.72$\times 10^{-1}$

- Recursive estimators: $\beta_{\text{opt}} \approx 0.55$ for SNR $= 5$dB; $\beta_{\text{opt}} \approx 0.28$ for SNR $= 10$dB

SNR	RWLS	RCML	RAMX	RMMSE
5 dB	7.02$\times 10^{-1}$	6.97$\times 10^{-1}$	6.99$\times 10^{-1}$	6.52$\times 10^{-1}$
10 dB	2.98$\times 10^{-1}$	2.98$\times 10^{-1}$	2.98$\times 10^{-1}$	2.94$\times 10^{-1}$

Summary

- Recursive RIR estimation with energy conservation constraint was studied
  - Constraint helps to improve performance
  - RMMSE estimator with uniform prior performs best

- To further improve performance: incorporate additional a priori information

Summary

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Thank you for your attention

Backup

Frequency Domain Representation

- The condition $\mathbf{\Theta}(n)^T\mathbf{\Theta}(n) - \mathbf{I} \preceq \mathbf{0}$ requires all eigenvalues of $\mathbf{\Theta}(n)^T\mathbf{\Theta}(n)$ to be smaller or equal to 1
- Let

$$r(n) = \begin{cases} \sum_{m=0}^{M-1-|n|} \theta_m(n)\theta_{|n|+m}(n) & |n| \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

denote the (unnormalized) auto-correlation function of the RIR $\underline{\theta}(n)$

- Then, the matrix $\mathbf{\Theta}(n)^T\mathbf{\Theta}(n) \in \mathbb{R}^{S\times S}$ can be written as a symmetric Toeplitz matrix where the first row is given by $r(n)$ for $n = 0, \ldots, S-1$, i.e. $\mathbf{\Theta}(n)^T\mathbf{\Theta}(n)$ is bandlimited
- Using the asymptotic equivalence of the eigenvalues of a bandlimited Toeplitz matrix and the corresponding circulant matrix, it follows

$$|\Theta(\omega, n)|^2 = \left| \sum_{m=0}^{M-1} \theta_m(n) e^{-j\omega m} \right|^2 \leq 1 \;\forall\, \omega \in [0, 2\pi)$$

Affine Minimax Estimator for RIR

- **Epigraphic form of the AMX optimization problem**:

$$\min_{\mathbf{M},\underline{u},\tau} \ \tau$$

subject to

$$\begin{bmatrix} \underline{\theta} \\ 1 \end{bmatrix}^T \begin{bmatrix} -\mathbf{M}^T\mathbf{M} & -\mathbf{M}^T\underline{u} \\ -\underline{u}^T\mathbf{M} & \tau - \mathrm{tr}\left\{(\mathbf{I}+\mathbf{M})\mathbf{R}_n^{-1}(\mathbf{I}+\mathbf{M})^T\right\} - \underline{u}^T\underline{u} \end{bmatrix} \begin{bmatrix} \underline{\theta} \\ 1 \end{bmatrix} \geq 0 \ \forall \ \underline{\theta} \in \Theta$$

- **S-procedure**: sufficient condition for the statement

for all $\underline{z}$: $\underline{z}^T\mathbf{F}_0\underline{z} \geq 0$, . . . , $\underline{z}^T\mathbf{F}_{\tilde{L}}\underline{z} \geq 0 \Rightarrow \underline{z}^T\mathbf{G}\underline{z} \geq 0$

to be true is the existence of $\lambda_0, \ldots, \lambda_{\tilde{L}} \geq 0$ such that $\mathbf{G} \succeq \lambda_0\mathbf{F}_0 + \cdots + \lambda_{\tilde{L}}\mathbf{F}_{\tilde{L}}$.

Affine Minimax Estimator for RIR (II)

- **Final optimization problem**:

$$\min_{\substack{\lambda_0 \geq 0, \ldots, \lambda_{\tilde{L}} \geq 0 \\ \tau, \alpha, x}} \tau$$

subject to

$$\begin{aligned} \lambda_l \geq x/L \quad (l = 0, \ldots, \tilde{L}), \\ \begin{bmatrix} x & \alpha \\ \alpha & 1 \end{bmatrix} \succeq \mathbf{0}, \\ \tau - (1 + 2\alpha + x)\,\mathrm{tr}\{\mathbf{R}_n^{-1}\} \geq \lambda_0 + \cdots + \lambda_{\tilde{L}}. \end{aligned}$$

Bayesian Sufficient Statistics

- **Definition**: (introduced by Kolmogorov)
  - $\underline{t}(\underline{x})$ is Bayesian sufficient statistic if for all priors $p(\underline{\theta})$ the a posteriori density $p(\underline{\theta}|\underline{x})$ can be written as $p(\underline{\theta}|\underline{x}) = p(\underline{\theta}|\underline{t}(\underline{x}))$.

    $\Rightarrow$ Given $\underline{t}(\underline{x})$, no additional information can be infered from $\underline{x}$

- Concept directly related to classical sufficiency:
  - using Fisher-Neyman factorization theorem: Classical sufficiency implies Bayesian sufficiency

$$p(\underline{\theta}|\underline{x}) = \frac{p(\underline{\theta})p(\underline{x}|\underline{\theta})}{\int p(\underline{\theta})p(\underline{x}|\underline{\theta})d\theta} = \frac{p(\underline{\theta})g(\underline{t}(\underline{x}),\underline{\theta})}{\int p(\underline{\theta})g(\underline{t}(\underline{x}),\underline{\theta})d\theta} = p(\underline{\theta}|\underline{t}(\underline{x}))$$

  - converse is also true under some regularity conditions