When to Add a New Dimension?

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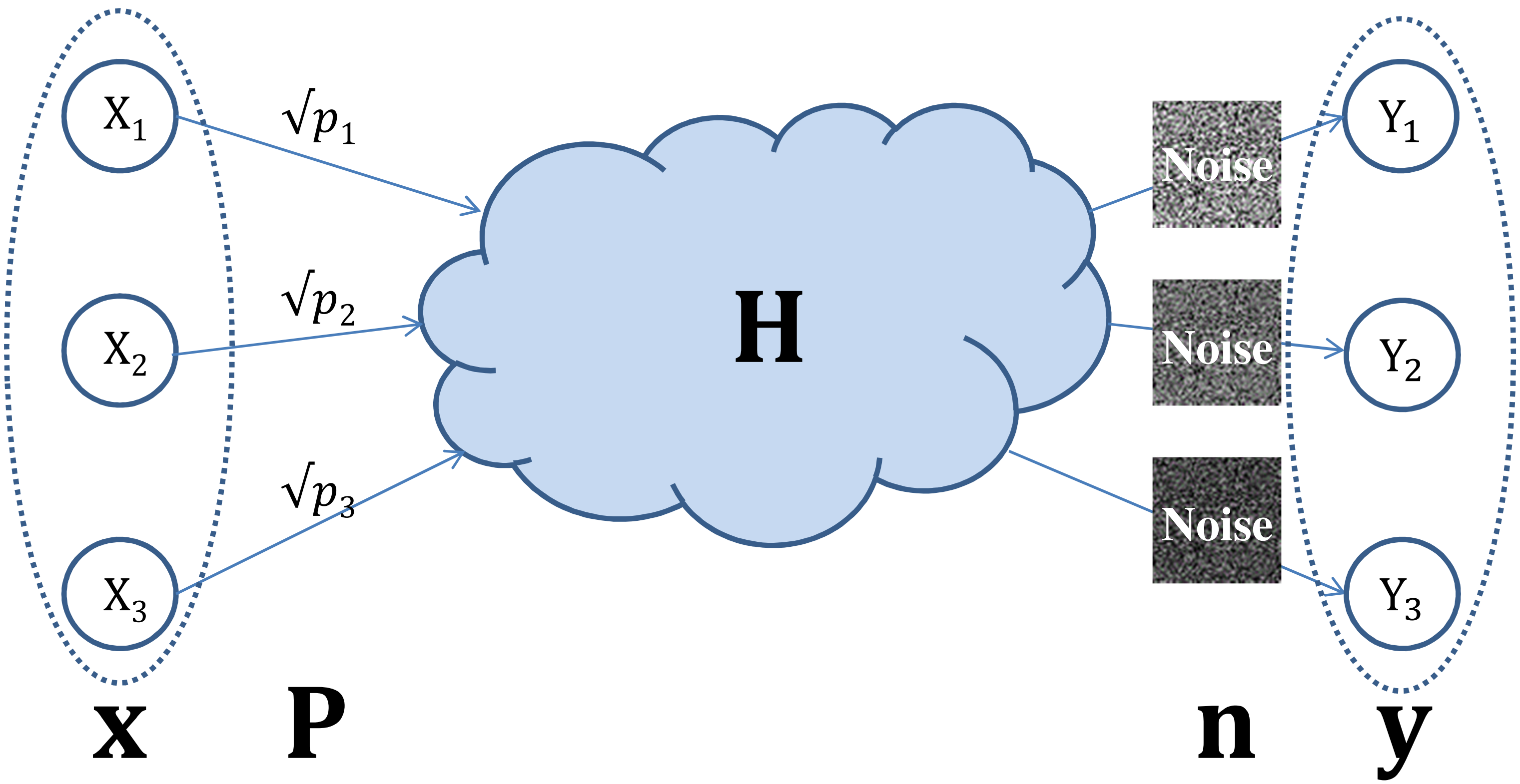
MIMO Gaussian Channel



- Multitone transmission (DSL, OFDM)
- Multiantenna transmission
- Fading channels (parallel *in time*)

General Problem

Model: $\mathbf{y} = \mathbf{HPx} + \mathbf{n}$

Objective function: $\mathcal{O}(\mathbf{x}, \mathbf{y}|\mathbf{H}, \mathbf{P})$

Generic optimization problem:

$$\begin{aligned} \underset{\mathbf{P}}{\text{optimize}} \quad & \mathcal{O}(\mathbf{x}, \mathbf{y}|\mathbf{H}, \mathbf{P}) && \text{Objective function} \\ \text{subject to} \quad & \mathbf{n} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) && \text{Gaussian noise} \\ & \mathrm{Tr}[\mathbf{Pxx}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}] \leq \rho = \mathrm{SNR} && \text{Total power constraint} \\ & \mathbf{H} \in \mathcal{C}_{\mathbf{H}}, \mathbf{P} \in \mathcal{C}_{\mathbf{P}}, \mathbf{x} \in \mathcal{C}_{\mathbf{x}} \cap \mathcal{X} && \text{Other constraints} \end{aligned}$$

$\mathbf{P}$: precoding matrix, $\mathbf{H}$: channel matrix, $\mathcal{C}$: constraint sets, $\mathcal{X}$: input alphabet, SNR: signal-to-noise ratio, $\mathbf{n}$: noise, $\mathcal{N}$: Gaussian, $\mathrm{Tr}[\cdot]$: trace, $(\cdot)^{\mathrm{T}}$: transpose, $\mathbf{I}$: identity matrix, $\mathbf{0}$: zero vector

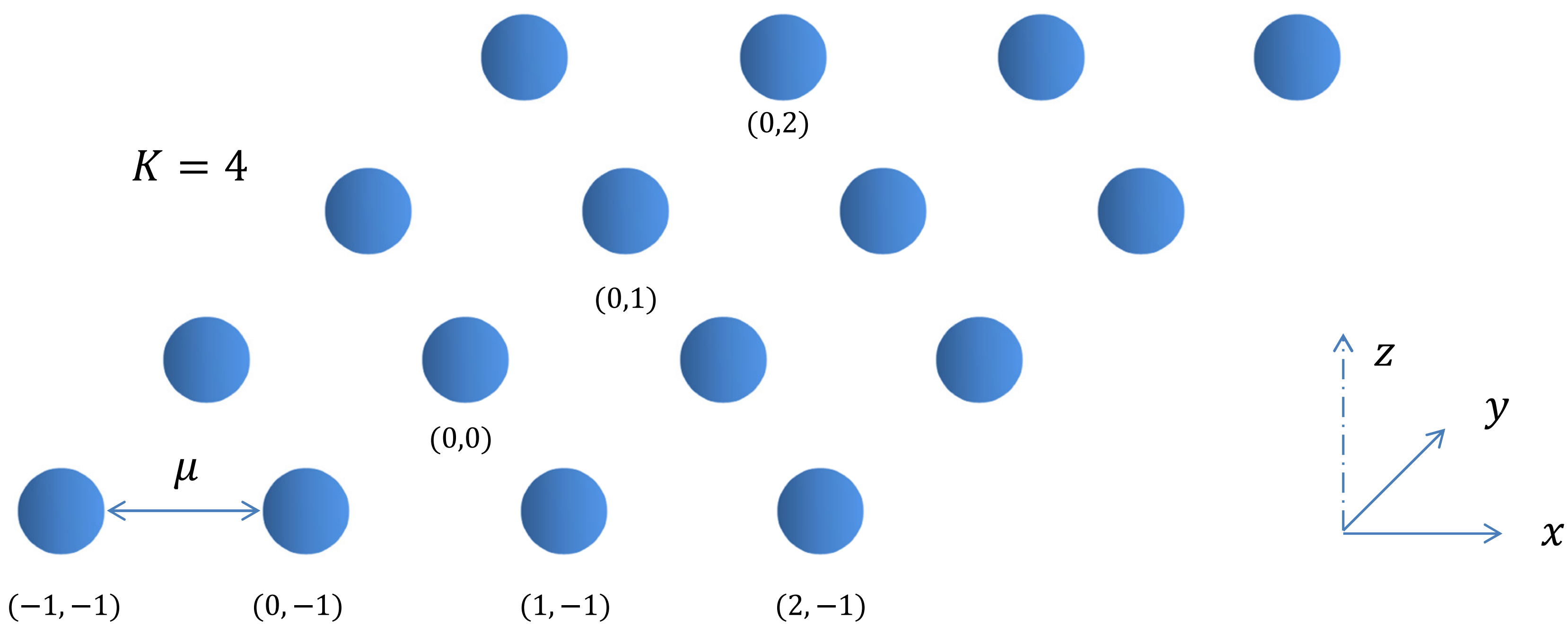
Outline

1. The Input
2. The Strategy
3. The Results

The Input

Lattices

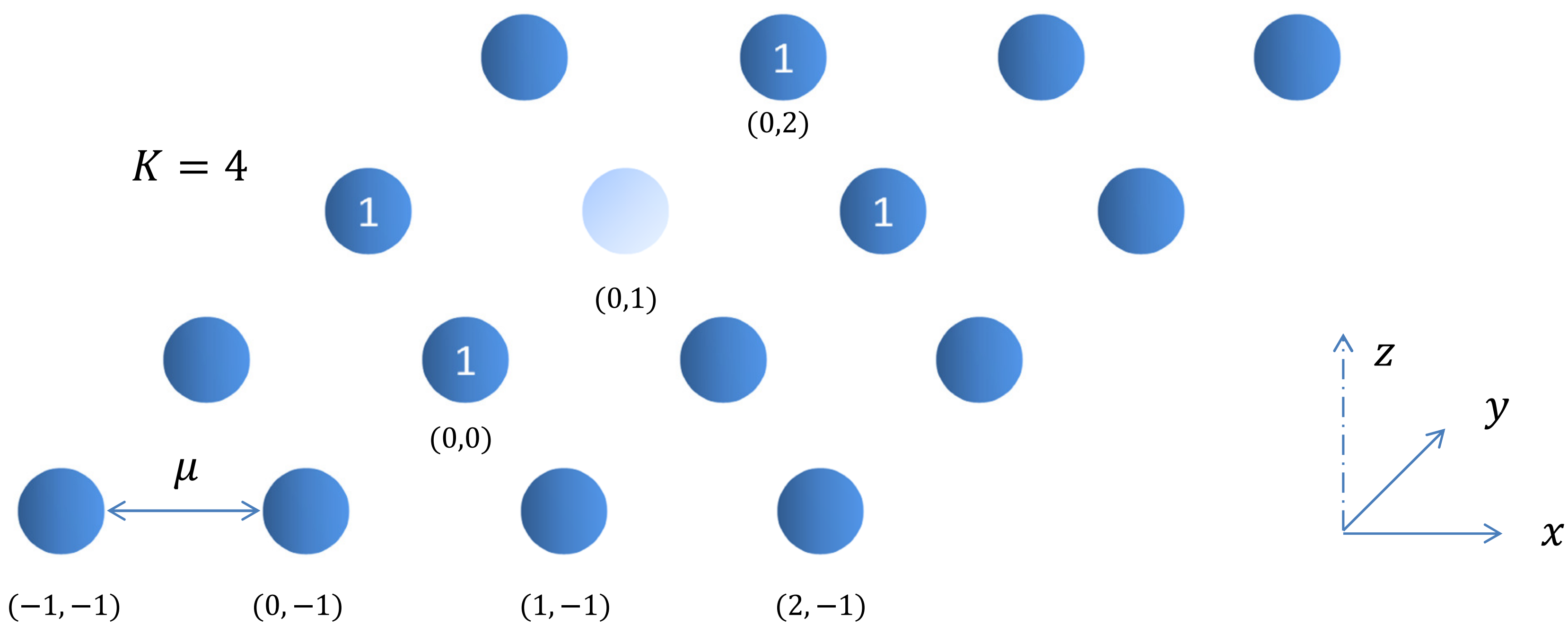
Lattices



Definition: An m-dimensional lattice Λ in $\mathbb{R}^n$ is the additive group of all integer linear combinations of m row vectors $v_1, v_2, \dots, v_m$, that are linearly independent over $\mathbb{R}$.

μ: minimum distance, K: kissing number (number of nearest neighbors)

Lattices



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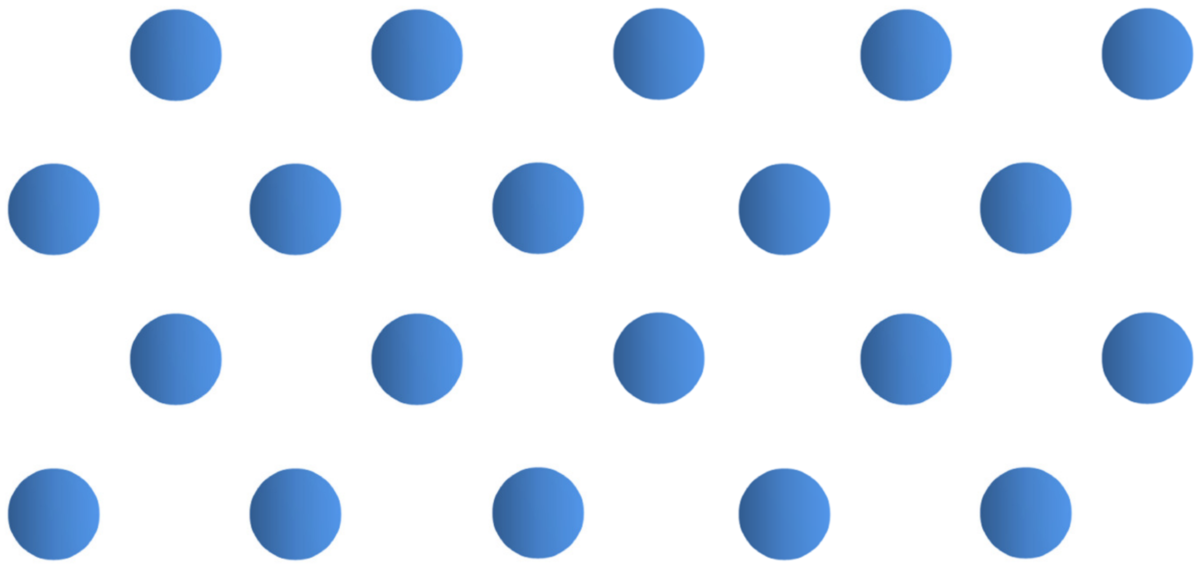

$K = 6$

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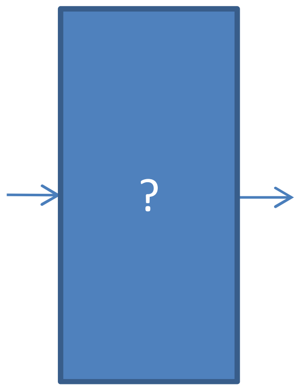
Why Lattices?

- Tradition
- Easy addressing
- Easy decoding
- Capacity achieving

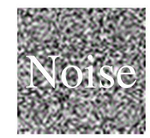
4×4

256 points
$\mathbb{Z}^4$ lattice
4 independent PAM



P

$$\begin{pmatrix} 1.44 & 0.22 & 0 & 0 \\ -0.83 & 0.37 & 0 & 0 \\ 0 & 0 & 0.29 & 0 \\ 0 & 0 & 0 & 0.12 \end{pmatrix} +$$



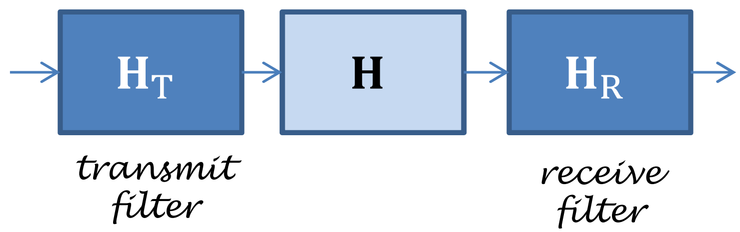
H

The Strategy

Divide and Conquer

Divide and Conquer

$$\mathbf{H} = \mathbf{U}_{\mathrm{H}}\boldsymbol{\Lambda}_{\mathrm{H}}\mathbf{V}_{\mathrm{H}}^{\mathrm{T}}$$



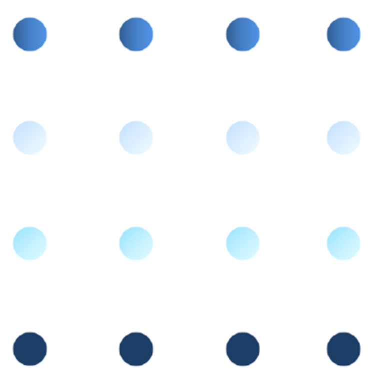
Parallelizing the channels

$$\mathbf{H}_{\mathrm{T}} = \mathbf{V}_{\mathrm{H}}\boldsymbol{\Lambda}_{\mathrm{H}_{\mathrm{T}}} \qquad \mathbf{H}_{\mathrm{R}} = \mathbf{U}_{\mathrm{H}}^{\mathrm{T}}$$

Divide and Conquer

- Capacity achieving
- "For the Gaussian signaling case and the low SNR regime, the dependence of the mutual information on the right singular vectors vanishes, making the optimal precoder design problem easy to solve." - Payaró, Palomar
- "For Schur-concave objective functions, the channel-diagonalizing structure is always optimal, whereas for Schur-convex functions, an optimal solution diagonalizes the channel only after a very specific rotation of the transmitted symbols." – Palomar, Cioffi, Lagunas
- Intuitive

4×4



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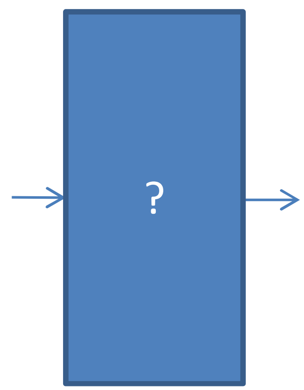

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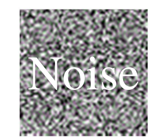
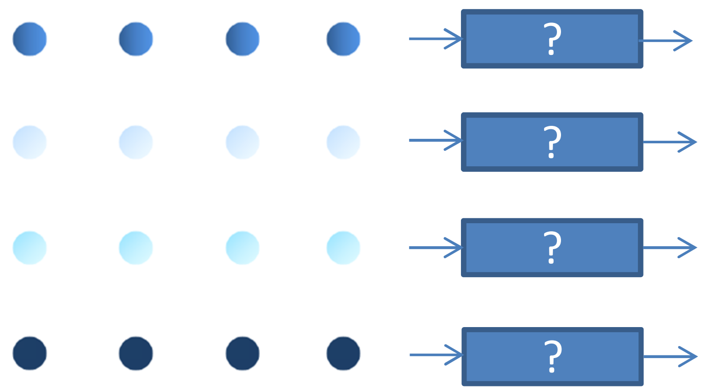


H

4×4



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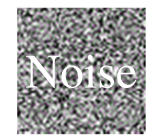


$\mathbf{P}$ $\mathbf{H}$

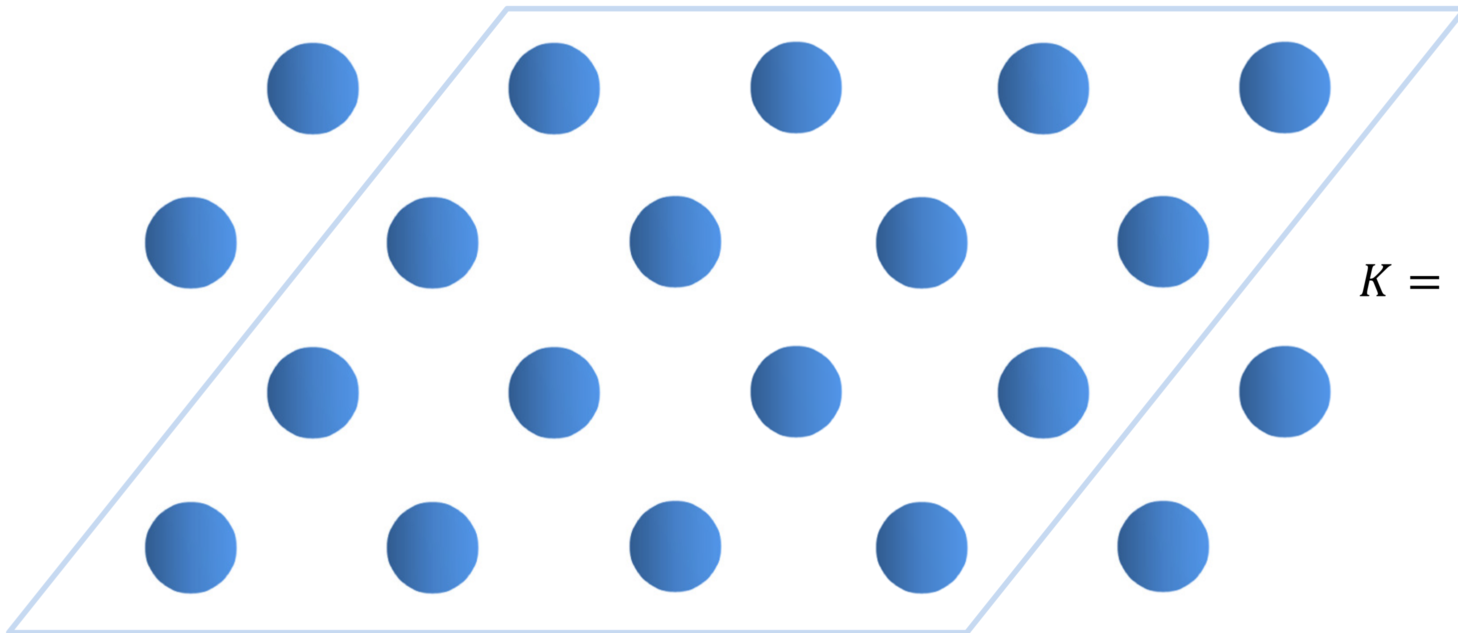
256 points
$\mathbb{Z}^4$ lattice
4 independent PAM

- We start with probability of error using a maximum likelihood decoder:

$$\mathcal{O}(\mathbf{x}, \mathbf{y} | \mathbf{H}, \mathbf{P}) = P_e$$

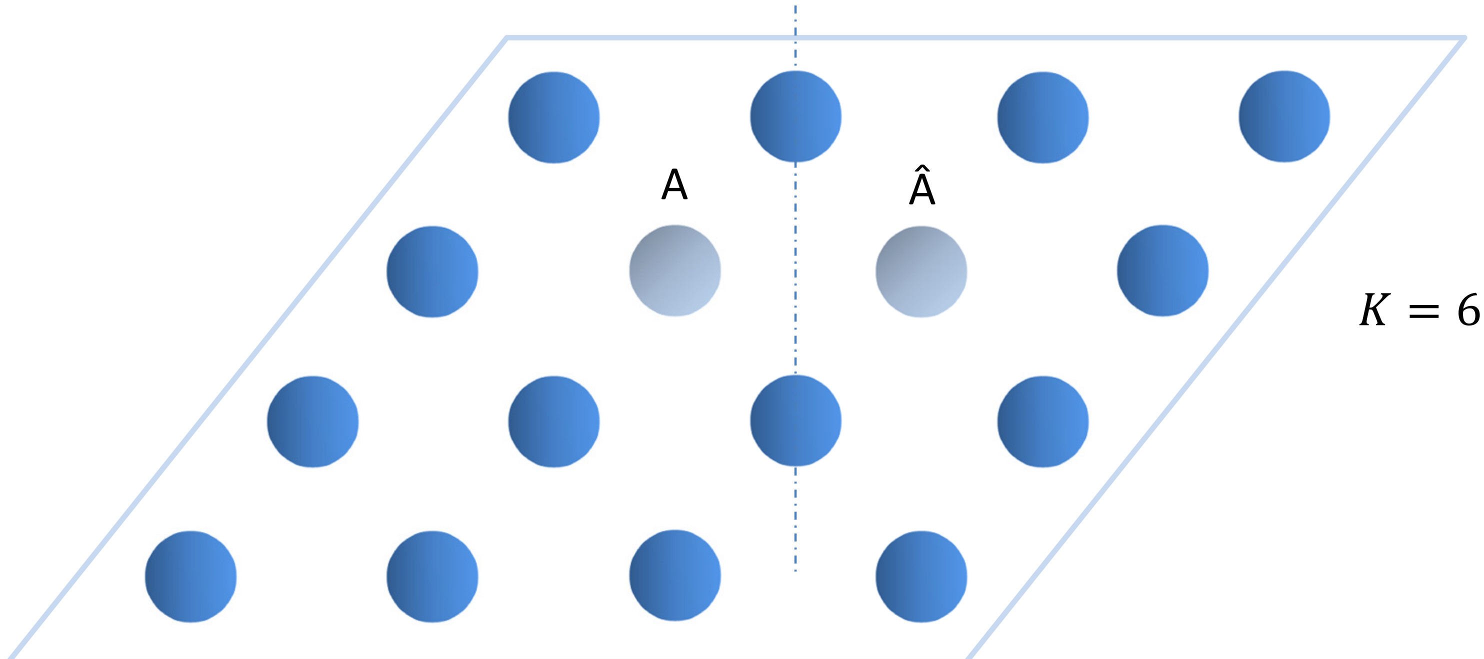
- Inputs normalized to unit variance in each dimension: $\mathbb{E}[\mathbf{x}\mathbf{x}^{\mathrm{T}}] = \mathbf{I}$, $\mathbb{E}[\mathbf{x}] = \mathbf{0}$

Divide and Conquer



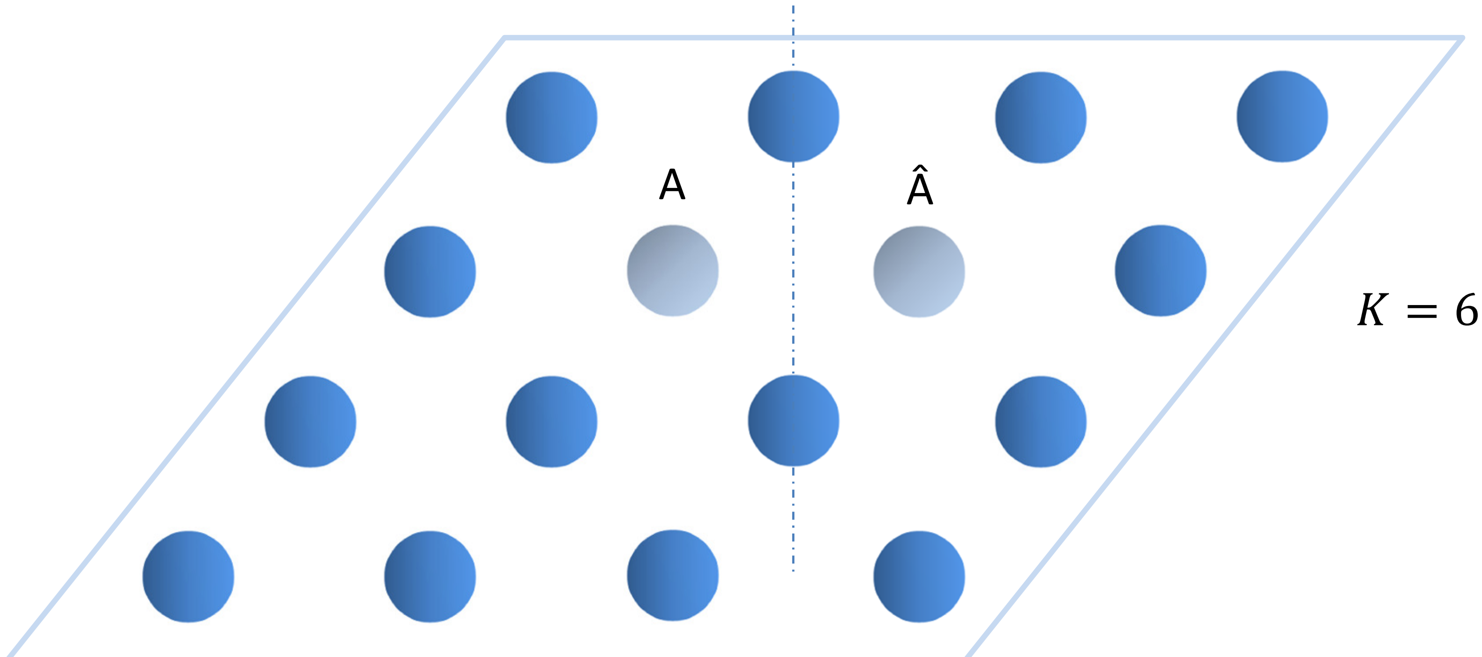
$K = 6$

Divide and Conquer



- Considering all pairs of points, we obtain an upper bound to the probability of error.

Divide and Conquer

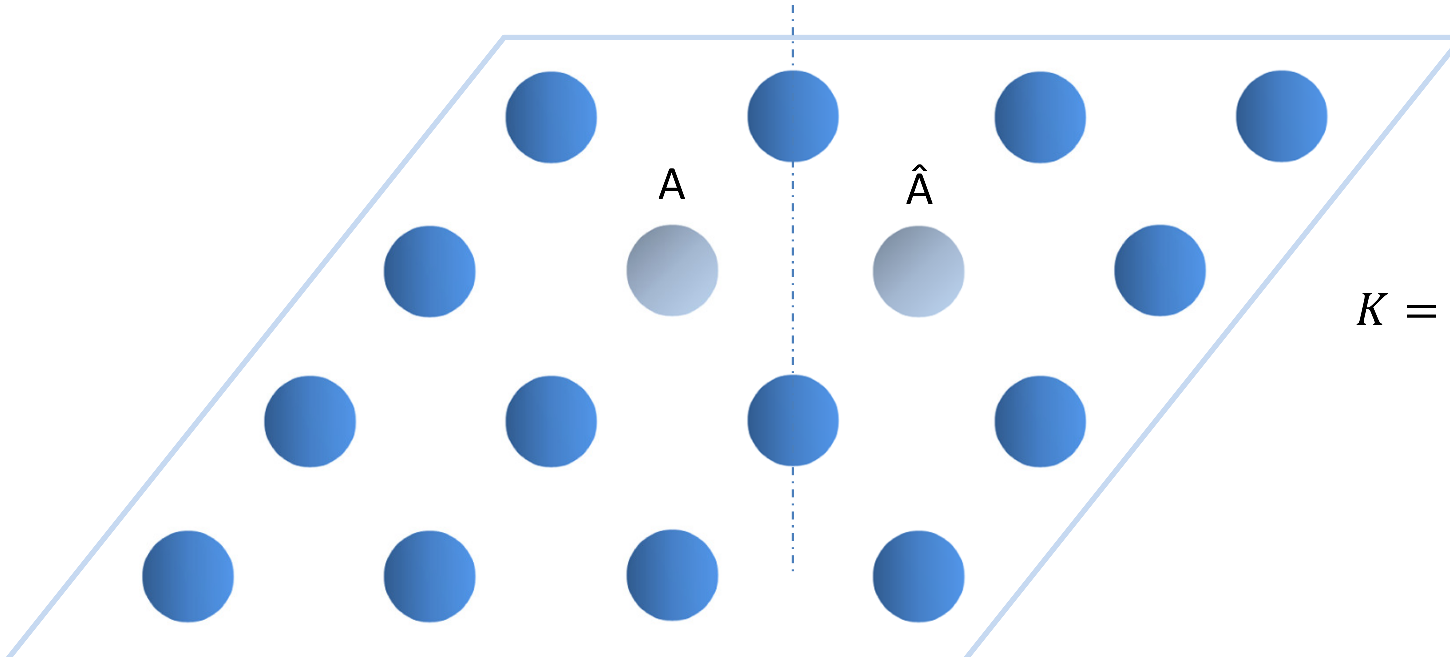


M_n *constellation size*

$$P_e \leq \sum_{n=1}^{N} \frac{K_n}{M_n} Q\left(\sqrt{-\mathrm{SNR}\frac{p_n|h_n|^2\mu_n^2}{4}}\right)$$

- Considering all pairs of points at minimum distance, we obtain a tighter upper bound, in several cases of interest.

Divide and Conquer



$K = 6$

M_n constellation size

upper bound on P_e

$$P_e \leq \frac{1}{2} \sum_{n=1}^{N} \frac{K_n}{M_n} \exp\left(-\mathrm{SNR}\frac{p_n |h_n|^2 \mu_n^2}{8}\right) = P_{e,UB}$$

- Considering all pairs of points at minimum distance, we obtain a tighter upper bound, in several cases of interest. K_n is the number of such pairs.

Divide and Conquer

- Bound becomes tight towards high SNR.
- Problem is converted to a convex optimization problem.
- Thresholds (SNR) where the channels are first allocated power are simpler to express, in terms of lattice parameters.
- Relationship with mutual information.

Problem Statement

$$\underset{p_1,\ldots,p_n}{\text{minimize}} \qquad P_{e,UB} = \frac{1}{2}\sum_{n=1}^{N} \frac{K_n}{M_n} \exp\left(-\text{SNR}\frac{p_n |h_n|^2 \mu_n^2}{8}\right)$$

subject to:

$$\mathbb{E}\left\{\text{tr}\left(\mathbf{H}_{\text{T}}\mathbf{x}\mathbf{x}^{\text{T}}\mathbf{H}_{\text{T}}^{\text{T}}\right)\right\} = \sum_{n=1}^{N} p_n \leq 1$$

Total power constraint

$$p_n \geq 0, \qquad n = 1, \ldots, N$$

When to add a new dimension?

$$p_n^* = 0 \qquad\qquad a_n \mathrm{g} < \eta$$

$$p_n^* = \frac{8}{\mathrm{g}|h_n|^2\mu_n^2}\ln\left(\frac{1}{\eta}a_n\mathrm{g}\right) \qquad\qquad a_n\mathrm{g} \geq \eta$$

Channel strength metric $a_n = \frac{K_n}{M_n}\frac{|h_n|^2\mu_n^2}{8}$

$a_1 \geq a_2 \geq \ldots \geq a_N$

The i^{th} strongest channel turns on at SNR g_i,

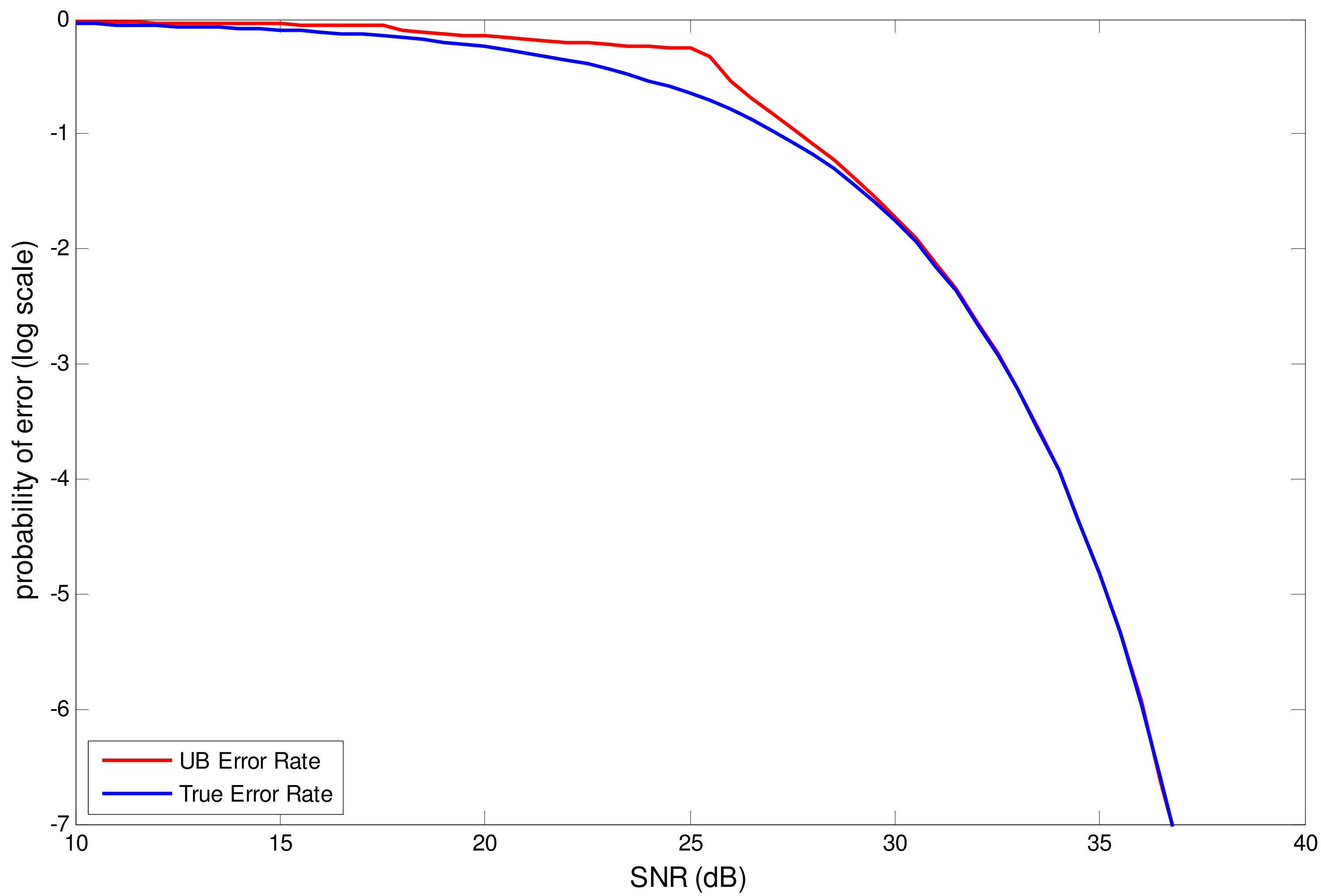
$$\mathrm{g}_i = \sum_{n=1}^{i-1}\left(\frac{8}{|h_n|^2\mu_n^2}\ln\frac{a_n}{a_i}\right)$$

The Results

Results

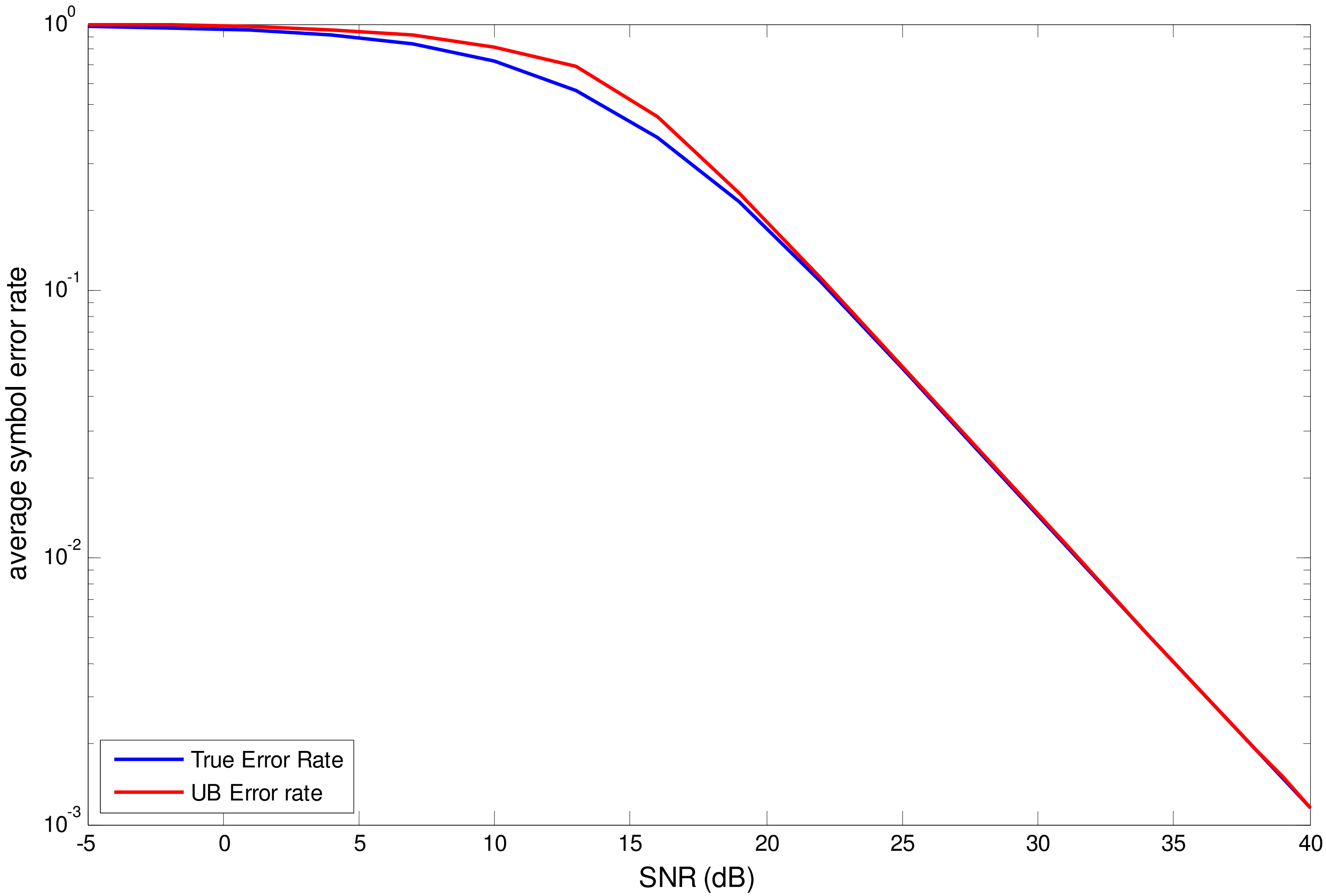
Error Performance

Comparison for filters which optimize P_e and $P_{e,UB}$



Error Performance

Average error rates for filters which optimize P_e and $P_{e,UB}$



Higher Dimensions

h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1
h_2	h_2	h_2	h_2	h_2	h_2	h_2	h_2	h_2	h_2
h_3	h_3	h_3	h_3	h_3	h_3	h_3	h_3	h_3	h_3
h_4	h_4	h_4	h_4	h_4	h_4	h_4	h_4	h_4	h_4

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}

- For a $2L$-dimensional lattice, we use L time slots.
- Typically, we choose $L \times N$ space-time slots (N lattice points) as one symbol or L time slots as one symbol (one lattice point).

Problem Statement

$$\underset{p_1,\ldots,p_n}{\text{minimize}} \quad P_{e,UB} = \frac{1}{2}\sum_{n=1}^{N}\frac{K_n}{M_n}\exp\left(-\text{SNR}\frac{p_n|h_n|^2\mu_n^2}{8}\right)$$

subject to:

$$\mathbb{E}\{\text{tr}(\mathbf{H}_\text{T}\mathbf{x}\mathbf{x}^\text{T}\mathbf{H}_\text{T}^\text{T})\} = \sum_{n=1}^{N} p_n \leq L$$ Total power constraint

$$p_n \geq 0, \qquad n = 1,\ldots,N$$

High SNR

- Power allocation tends to pre-equalization.

$$p_i^* h_i^2 \mu_i^2 = p_j^* h_j^2 \mu_j^2$$

- Each lattice “passes through”.

$$p_i^* = \frac{L}{h_i^2 \mu_i^2} \left(\sum_{n=1}^{N} \frac{1}{h_n^2 \mu_n^2} \right)^{-1}$$

$$\frac{1}{2} \sum_{n=1}^{N} \frac{K_n}{M_n} \exp\left(-\mathrm{SNR} \frac{L}{8} \left(\sum_{n=1}^{N} \frac{1}{h_n^2 \mu_n^2} \right)^{-1} \right)$$

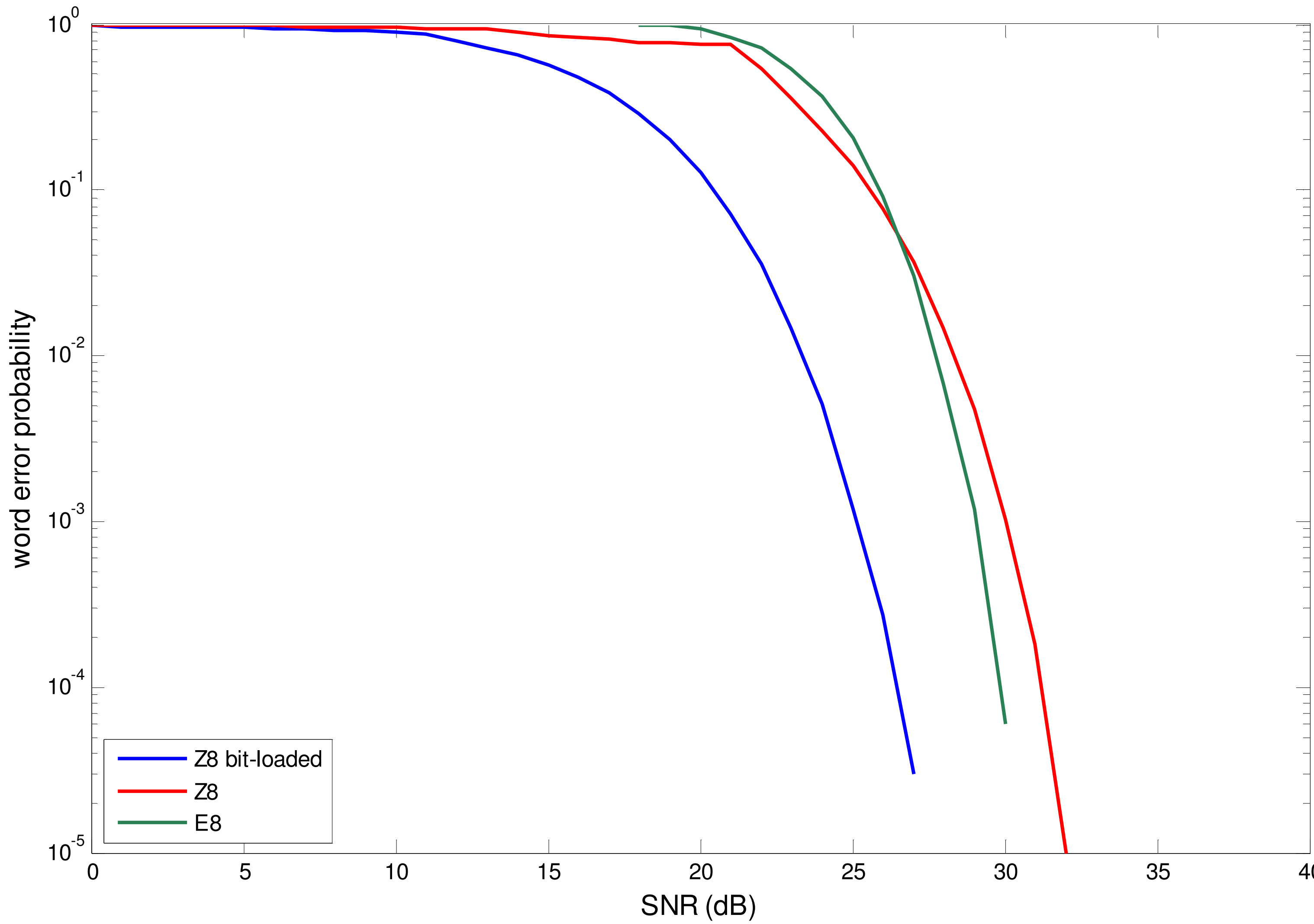
- The role of the harmonic mean

Extensibility

- Continuous approximation
- Bit-loading
  - Fisher & Huber
- Non-equiprobable signaling
  - Calderbank & Ozarow
- Constellation shaping

Extensibility

Word error probabilities for higher dimensional or bit-loaded lattices



Conclusions

- The divide-and-conquer technique along with the use of lattices is a simple yet effective way of transmitting information over a MIMO Gaussian channel.

- Due to the scalarization of the problem, the technique allows easy inclusion of other methods of power or error minimization, like bit-loading and shaping.

- Issues remaining to be dealt with
  - Coherence time and across space transmission
  - Addressing and decoding of large high-dimensional constellations

Thank you for attending!

Questions?

References

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Space versus Time

h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1
h_2	h_2	h_2	h_2	h_2	h_2	h_2	h_2	h_2	h_2
h_3	h_3	h_3	h_3	h_3	h_3	h_3	h_3	h_3	h_3
h_4	h_4	h_4	h_4	h_4	h_4	h_4	h_4	h_4	h_4

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}

h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1	h_1
h_2	h_2	h_2	h_2	h_2	h_2	h_2	h_2	h_2	h_2
h_3	h_3	h_3	h_3	h_3	h_3	h_3	h_3	h_3	h_3
h_4	h_4	h_4	h_4	h_4	h_4	h_4	h_4	h_4	h_4