Probabilistic SINR Constrained Robust Transmit Beamforming:
A Bernstein-type Inequality Based Conservative Approach

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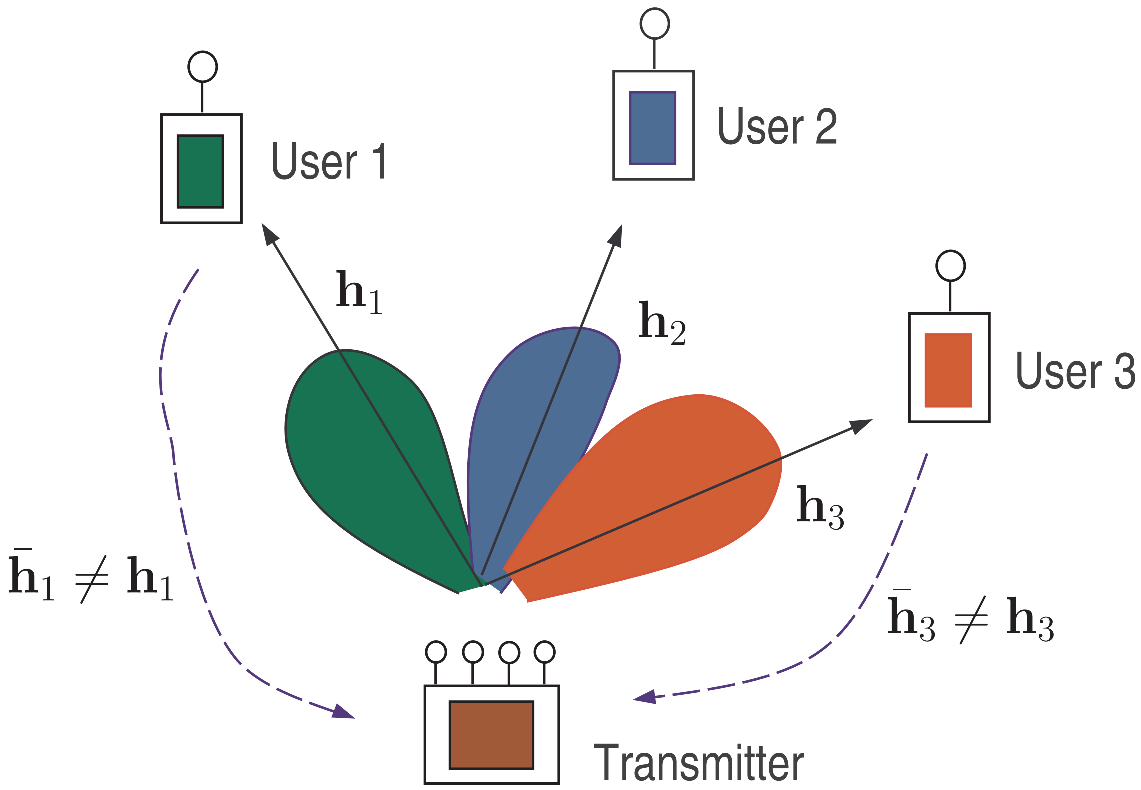
Signal Model and Problem Statement

Scenario: multiuser downlink, by transmit beamforming:



Goal: beamforming design that is 'robust' against the imperfect channel state information (CSI) at the transmitter.

Signal Model and Problem Statement (cont')

- The transmitter has N_t antennas; there are K single-antenna users.
- Let $\mathbf{h}_i \in \mathbb{C}^{N_t}$ be user i's channel vector, and $\boldsymbol{w}_i \in \mathbb{C}^{N_t}$ be user i's beamformer, for $i = 1, \ldots, K$.
- The SINR of user i:

$$\mathrm{SINR}_i = \frac{|\mathbf{h}_i^H \boldsymbol{w}_i|^2}{\sum_{k \neq i}^{K} |\mathbf{h}_i^H \boldsymbol{w}_k|^2 + \sigma_i^2},$$

where $\sigma_i^2 > 0$ is the noise power of user i.

Signal Model and Problem Statement (cont')

- A design formulation with perfect CSI:

$$\min_{\substack{\boldsymbol{w}_i \in \mathbb{C}^{N_t}, \\ i=1,\ldots,K}} \quad \sum_{i=1}^{K} \|\boldsymbol{w}_i\|^2 \tag{1a}$$

$$\text{s.t.} \quad \frac{|\mathbf{h}_i^H \boldsymbol{w}_i|^2}{\sum_{k\neq i}^{K} |\mathbf{h}_i^H \boldsymbol{w}_k|^2 + \sigma_i^2} \geq \gamma_i, \quad i = 1, \ldots, K, \tag{1b}$$

where $\gamma_i > 0$ stands for the preset target SINR value of user i.

- More than one way to solve problem (1) [Farrokhi99,Bengtsson01]:
  - uplink-downlink duality
  - semidefinite relaxation (SDR)
  - second-order cone program (SOCP) reformulation

[Farrokhi99] F. Rashid-Farrokhi, K. J. R. Liu, and L. Tassiulas, "Transmit beamforming and power control for cellular wireless systems," *IEEE J. Sel. Areas Commun.*, vol. 16, no. 8, pp. 1437-1450, Oct. 1999.

[Bengtsson01] M. Bengtsson and B. Ottersten, "Optimal and suboptimal transmit beamforming," *Chapter 18 in Handbook of Antennas in Wireless Communications*, L. C. Godara, Ed., CRC Press, Aug. 2001.

Signal Model and Problem Statement (cont')

- Due to imperfect channel estimation and limited feedback, the transmitter may only have inaccurate CSI in practice.
- Let $\bar{\mathbf{h}}_1, \ldots, \bar{\mathbf{h}}_K \in \mathbb{C}^{N_t}$ denote the channel estimates at the transmitter. The true channels can be expressed as

$$\mathbf{h}_i = \bar{\mathbf{h}}_i + \mathbf{e}_i, \ i = 1, \ldots, K,$$

  where $\mathbf{e}_i \in \mathbb{C}^{N_t}$ represents the CSI error vector.
- We assume:

$$\mathbf{e}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_i), \ i = 1, \ldots, K,$$

  where $\mathbf{C}_i \succeq \mathbf{0}$.

Signal Model and Problem Statement (cont')

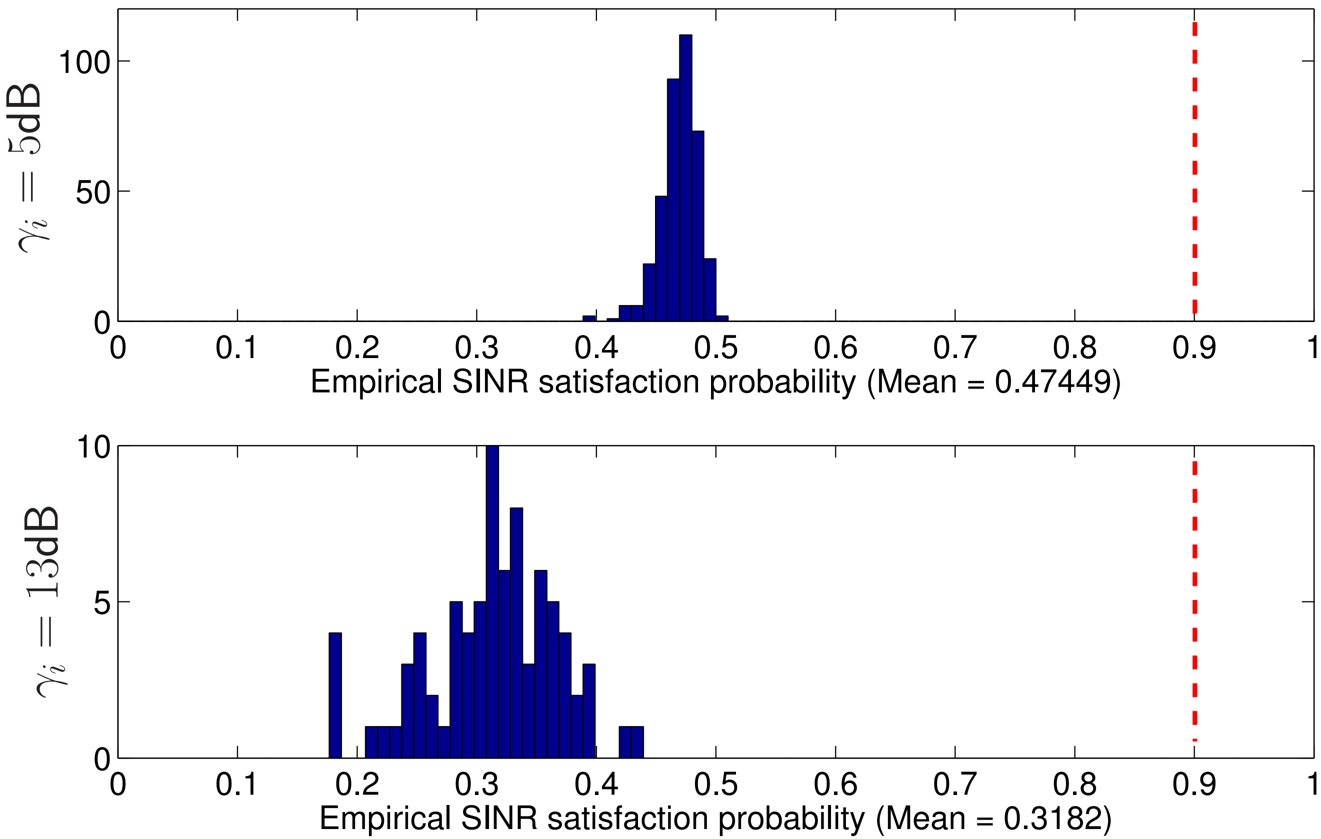
- It is desirable to design the beamformers $\{\boldsymbol{w}_i\}_{i=1}^K$ such that the SINR satisfaction probability

$$\Pr_{\mathbf{e}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_i)} \left\{ \frac{|(\bar{\mathbf{h}}_i + \mathbf{e}_i)^H \boldsymbol{w}_i|^2}{\sum_{k \neq i}^K |(\bar{\mathbf{h}}_i + \mathbf{e}_i)^H \boldsymbol{w}_k|^2 + \sigma_i^2} \geq \gamma_i \right\}$$

is close to one.

Performance of Non-robust Design

Simulation results of histogram of SINR satisfaction probability (of user 1) for $N_t = K = 3$, $\mathbf{C}_i = 0.002\mathbf{I}_{N_t}$ and $\sigma_i^2 = 0.01$.



500 channel realizations of $\{\bar{\mathbf{h}}_i\}_{i=1}^3$ are tested. The empirical probability is obtained by averaging over 10,000 Gaussian CSI errors.

Probabilistic SINR Constrained Robust Design

- Let $\rho_i \in (0,1]$ denotes user i's maximum tolerable SINR outage probability.
- An outage constrained robust design formulation [Shenouda08]:

$$\min_{\substack{\boldsymbol{w}_i \in \mathbb{C}^{N_t},\\ i=1,\ldots,K}} \sum_{i=1}^{K} \|\boldsymbol{w}_i\|^2 \tag{2a}$$

$$\text{s.t. } \Pr_{\mathbf{e}_i \sim \mathcal{CN}(\mathbf{0},\mathbf{C}_i)} \left\{ \frac{|(\bar{\mathbf{h}}_i + \mathbf{e}_i)^H \boldsymbol{w}_i|^2}{\sum_{k\neq i}^{K} |(\bar{\mathbf{h}}_i + \mathbf{e}_i)^H \boldsymbol{w}_k|^2 + \sigma_i^2} \geq \gamma_i \right\} \geq 1 - \rho_i,$$

$$i = 1, \ldots, K. \tag{2b}$$

[Shenouda08] M. B. Shenouda and T. N. Davidson, Probabilistically-constrained approaches to the design of the multiple antenna downlink, in *Proc. 42nd Asilomar Conference*, Pacific Grove, Oct. 26-29, 2008, pp. 1120-1124.

Probabilistic SINR Constrained Robust Design (cont')

- Solving problem (2) is challenging because

$$\Pr\left\{(\bar{\mathbf{h}}_i + \mathbf{e}_i)^H \left(\frac{1}{\gamma_i}\boldsymbol{w}_i\boldsymbol{w}_i^H - \sum_{k\neq i}\boldsymbol{w}_k\boldsymbol{w}_k^H\right)(\bar{\mathbf{h}}_i + \mathbf{e}_i) \geq \sigma_i^2\right\} \geq 1 - \rho_i$$

  - nonconvex SINR formulation
  - no closed-form expression for the probability function

- Existing approximation methods include
  - probabilistic SINR constrained SOC problem [Shenouda08];
  - worst-case robust beamforming problem [Wang10]

Conservative approximation method: the probabilistic SINR constraints are guaranteed to be satisfied.

[Wang10] K.-Y. Wang, T.-H. Chang, W.-K. Ma, and C.-Y. Chi, "A semidefinite relaxation based conservative approach to robust transmit beamforming with probabilistic SINR constraints," in *Proc. EUSIPCO*, Aalborg, Denmark, August 23-27, 2010, pp. 407-411.

Semidefinite relaxation (SDR)

- Replace each $\boldsymbol{w}_i \boldsymbol{w}_i^H$ with $\mathbf{W}_i \succeq \mathbf{0}$:

$$\min_{\substack{\mathbf{W}_i \in \mathbb{H}^{N_t} \\ i=1,\ldots,K}} \sum_{i=1}^{K} \mathrm{Tr}(\mathbf{W}_i) \tag{3a}$$

$$\text{s.t. } \Pr\left\{ (\bar{\mathbf{h}}_i + \mathbf{e}_i)^H \left(\frac{1}{\gamma_i} \mathbf{W}_i - \sum_{k \neq i} \mathbf{W}_k \right) (\bar{\mathbf{h}}_i + \mathbf{e}_i) \geq \sigma_i^2 \right\} \geq 1 - \rho_i,$$

$$i = 1, \ldots, K, \tag{3b}$$

$$\mathbf{W}_1, \ldots, \mathbf{W}_K \succeq \mathbf{0} \tag{3c}$$

- Each probability constraint is of the form:

$$\Pr\{\mathbf{v}^H \mathbf{Q} \mathbf{v} + 2\mathrm{Re}\{\mathbf{v}^H \mathbf{r}\} + s \geq 0\} \geq 1 - \rho,$$

where $\mathbf{v} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_n)$.

[Luo10] Z.-Q. Luo, W.-K. Ma, A. M.-C. So, Y. Ye, and S. Zhang, "Semidefinite relaxation of quadratic optimization problems, *IEEE Signal Process. Mag.*, pp. 20-34, May 2010.

Approximation by Bernstein-type Inequality

- **Lemma 1** *Let* $\mathbf{v} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_n)$, $\mathbf{Q} \in \mathbb{H}^n$, $\mathbf{r} \in \mathbb{C}^n$ *and* $s \in \mathbb{C}$. *Then for any* $\rho \in [0,1]$, *we have*

$$\Pr\left\{\mathbf{v}^H\mathbf{Q}\mathbf{v} + 2\mathrm{Re}\{\mathbf{v}^H\mathbf{r}\} + s \geq T\right\} \geq 1 - \rho, \quad (4)$$

where

$$T = \mathrm{Tr}(\mathbf{Q}) - \sqrt{2(-\ln(\rho))}\sqrt{\|\mathbf{Q}\|_F^2 + 2\|\mathbf{r}\|^2} + \ln(\rho)\ \lambda^+(\mathbf{Q}) + s, \quad (5)$$

in which $\lambda^+(\mathbf{Q}) = \max\{\lambda_{\max}(-\mathbf{Q}), 0\}$.

- Lemma 1 is obtained by extending the result in [Bechar10]. Inequality (4) is a Bernstein-type inequality.

[Bechar10] I. Bechar, "A Bernstein-type inequality for stochastic processes of quadratic forms of gaussian variables," available on http://arxiv.org/abs/0909.3595.

Approximation by Bernstein-type Inequality (cont')

- By Lemma 1, we obtain that

$$\mathrm{Tr}(\mathbf{Q}) - \sqrt{2(-\ln(\rho))}\sqrt{\|\mathbf{Q}\|_F^2 + 2\|\mathbf{r}\|^2} + \ln(\rho)\ \lambda^+(\mathbf{Q}) + s \geq 0 \tag{6}$$

is a sufficient condition for achieving

$$\Pr\{\mathbf{v}^H\mathbf{Q}\mathbf{v} + 2\mathrm{Re}\{\mathbf{v}^H\mathbf{r}\} + s \geq 0\} \geq 1 - \rho.$$

Approximation by Bernstein-type Inequality (cont')

- **Observation:** (6) can be reformulated as

$$\mathrm{Tr}\,(\mathbf{Q}) - \sqrt{2\delta}x - \delta y + s \geq 0, \tag{7a}$$

$$\sqrt{\|\mathbf{Q}\|_F^2 + 2\|\mathbf{r}\|^2} \leq x, \tag{7b}$$

$$y\mathbf{I}_n + \mathbf{Q} \succeq \mathbf{0}, \tag{7c}$$

$$y \geq 0, \tag{7d}$$

where $x, y \in \mathbb{R}$ are slack (decision) variables.

- In summary, we can use (7) as a convex (conservative) approximation to

$$\Pr\{\mathbf{v}^H\mathbf{Q}\mathbf{v} + 2\mathrm{Re}\{\mathbf{v}^H\mathbf{r}\} + s \geq 0\} \geq 1 - \rho.$$

Proposed Approximation Formulation

- Define $\delta_i \triangleq -\ln(\rho_i),\ i = 1, \ldots, K,$ and apply (7) to (3):

$$
\begin{aligned}
\min_{\substack{\mathbf{W}_i \in \mathbb{H}^{N_t}, x_i, y_i \in \mathbb{R}, \\ i=1,\ldots,K}} \quad & \sum_{i=1}^{K} \mathrm{Tr}(\mathbf{W}_i) && (8) \\
\text{s.t. } & \mathrm{Tr}\left(\mathbf{Q}_i(\mathbf{W}_1, \ldots, \mathbf{W}_K)\right) - \sqrt{2\delta_i} x_i - \delta_i y_i \\
& \quad + s_i(\mathbf{W}_1, \ldots, \mathbf{W}_K) \geq 0,\ i = 1, \ldots, K, \\
& \left\| \begin{bmatrix} \mathrm{vec}\left(\mathbf{Q}_i(\mathbf{W}_1, \ldots, \mathbf{W}_K)\right) \\ \sqrt{2}\mathbf{r}_i(\mathbf{W}_1, \ldots, \mathbf{W}_K) \end{bmatrix} \right\| \leq x_i,\ i = 1, \ldots, K, \\
& y_i \mathbf{I} + \mathbf{Q}_i(\mathbf{W}_1, \ldots, \mathbf{W}_K) \succeq \mathbf{0},\ i = 1, \ldots, K, \\
& y_i \geq 0,\ \mathbf{W}_i \succeq \mathbf{0},\ i = 1, \ldots, K,
\end{aligned}
$$

is an convex approximation to problem (2).

Proposed Approximation Method (cont')

- The SDR problem (8) is in general an approximation because the associated optimal $\{\mathbf{W}_i\}_{i=1}^K$ may not be of rank one.
- If the optimal $\{\mathbf{W}_i\}_{i=1}^K$ of (8) is not of rank one, additional solution approximation procedures, e.g., Gaussian randomization, is needed [Luo10].
- In computer simulations, we found that it is very rare to obtain a higher-rank solution.

[Luo10] Z.-Q. Luo, W.-K. Ma, A. M.-C. So, Y. Ye, and S. Zhang, "Semidefinite relaxation of quadratic optimization problems, *IEEE Signal Process. Mag.*, pp. 20-34, May 2010.

Simulation Setting

- $N_t = 3$ and $K = 3$.
- 500 sets of channel estimates $\{\bar{\mathbf{h}}_i\}_{i=1}^{K}$ following the i.i.d. complex Gaussian distribution.
- $\rho_i = 0.1$ and $\sigma_i^2 = 0.01$ for all $i = 1, \ldots, K$.
- $\mathbf{C}_i = 0.002\mathbf{I}_3$ for all $i = 1, \ldots, K$.

Rank-one Solutions

We declare that the obtained solution $(\mathbf{W}_1, \ldots, \mathbf{W}_K)$ is of *rank one* if the following condition is satisfied

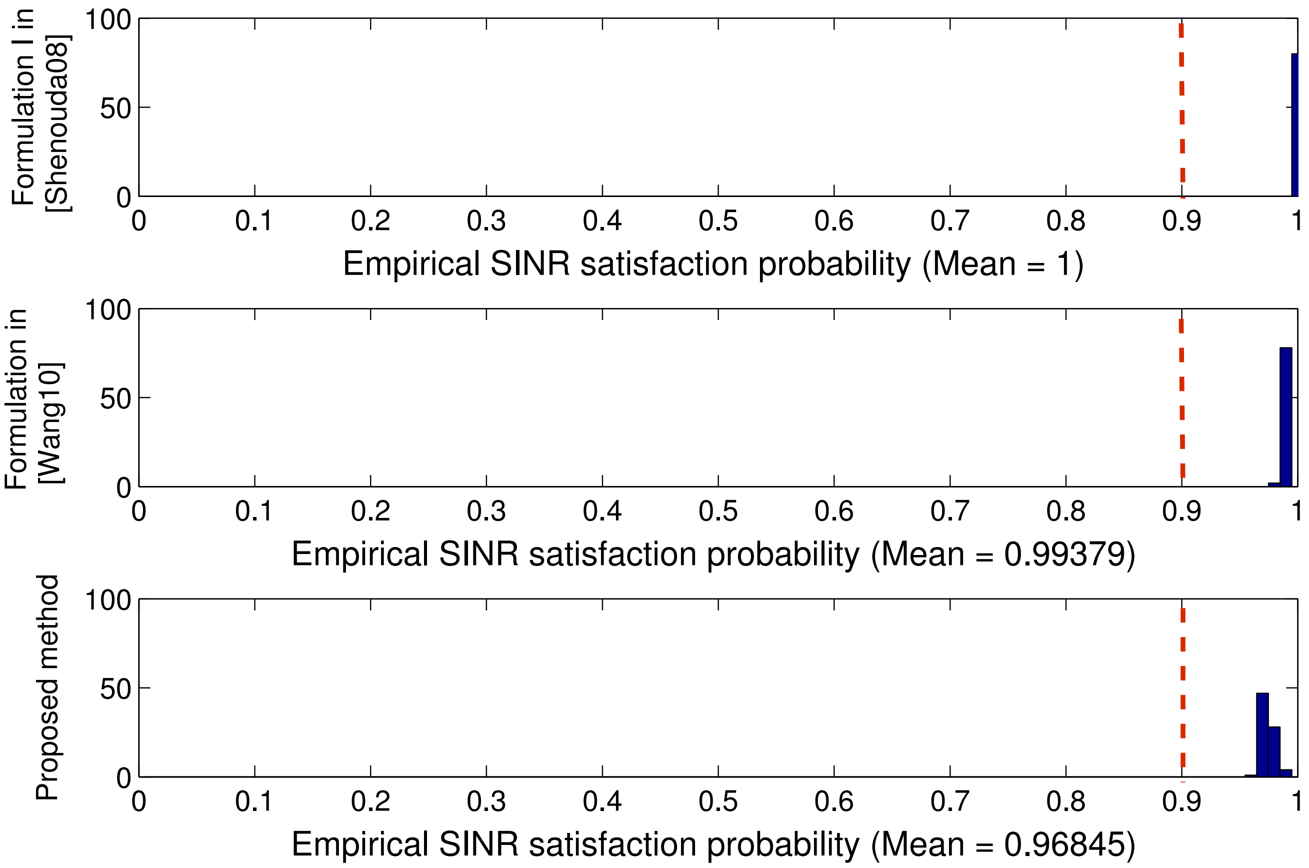
$$\frac{\lambda_{\max}(\mathbf{W}_i)}{\mathrm{Tr}(\mathbf{W}_i)} \geq 0.99 \text{ for all } i = 1, \ldots, K. \tag{9}$$

γ (dB)	1	3	5	7
$\rho_i = 0.1$	500/500	489/489	482/482	475/475
$\rho_i = 0.01$	499/499	479/480	475/475	463/463

γ (dB)	9	11	13	15
$\rho_i = 0.1$	462/462	441/441	419/419	363/363
$\rho_i = 0.01$	450/450	428/428	387/387	322/322

x/y: x is the number of realizations for which the rank-one solution is obtained, and y is the number of feasible channel realizations.

Histogram of SINR Satisfaction Probability ($\gamma_i = 13$dB)



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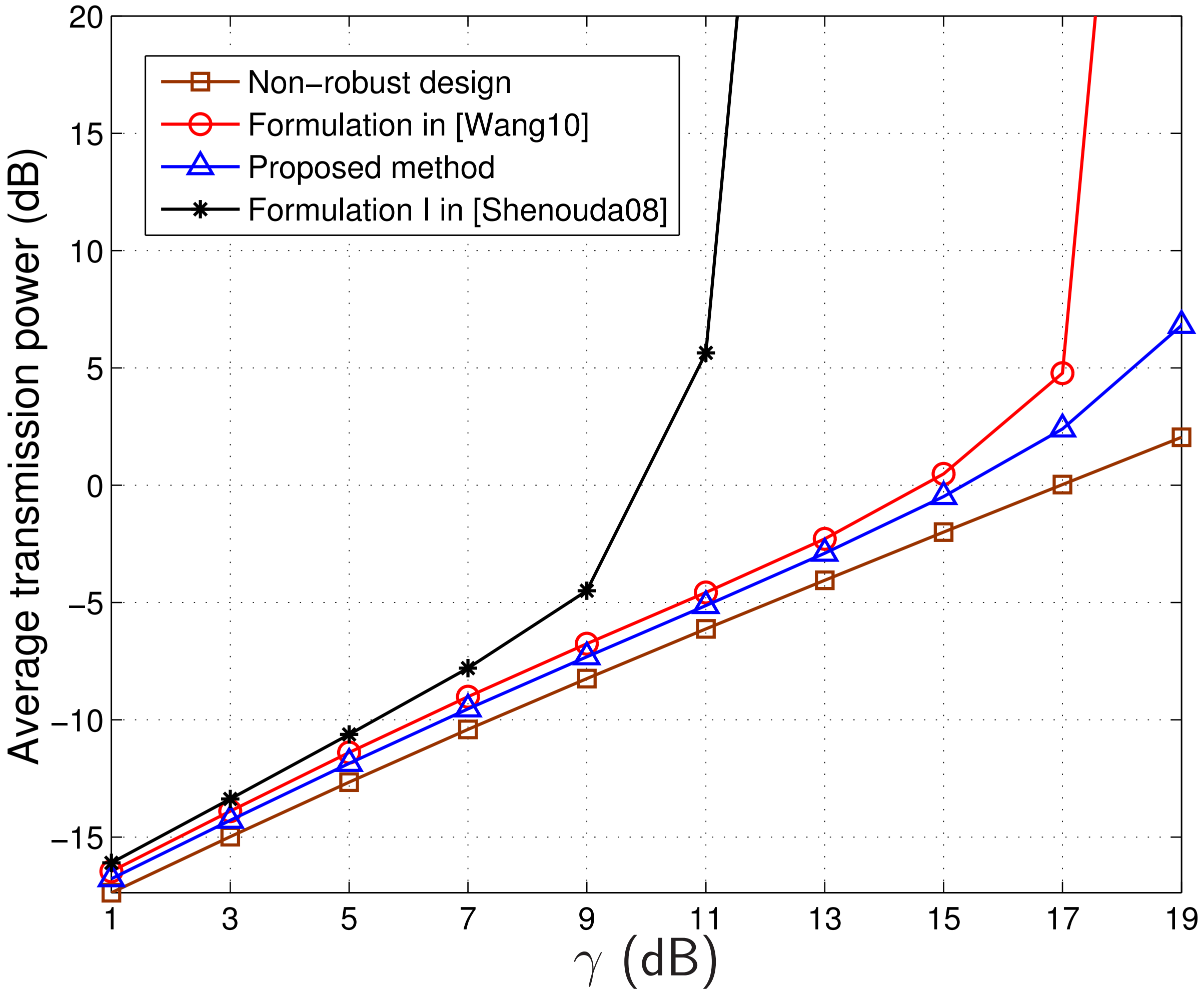
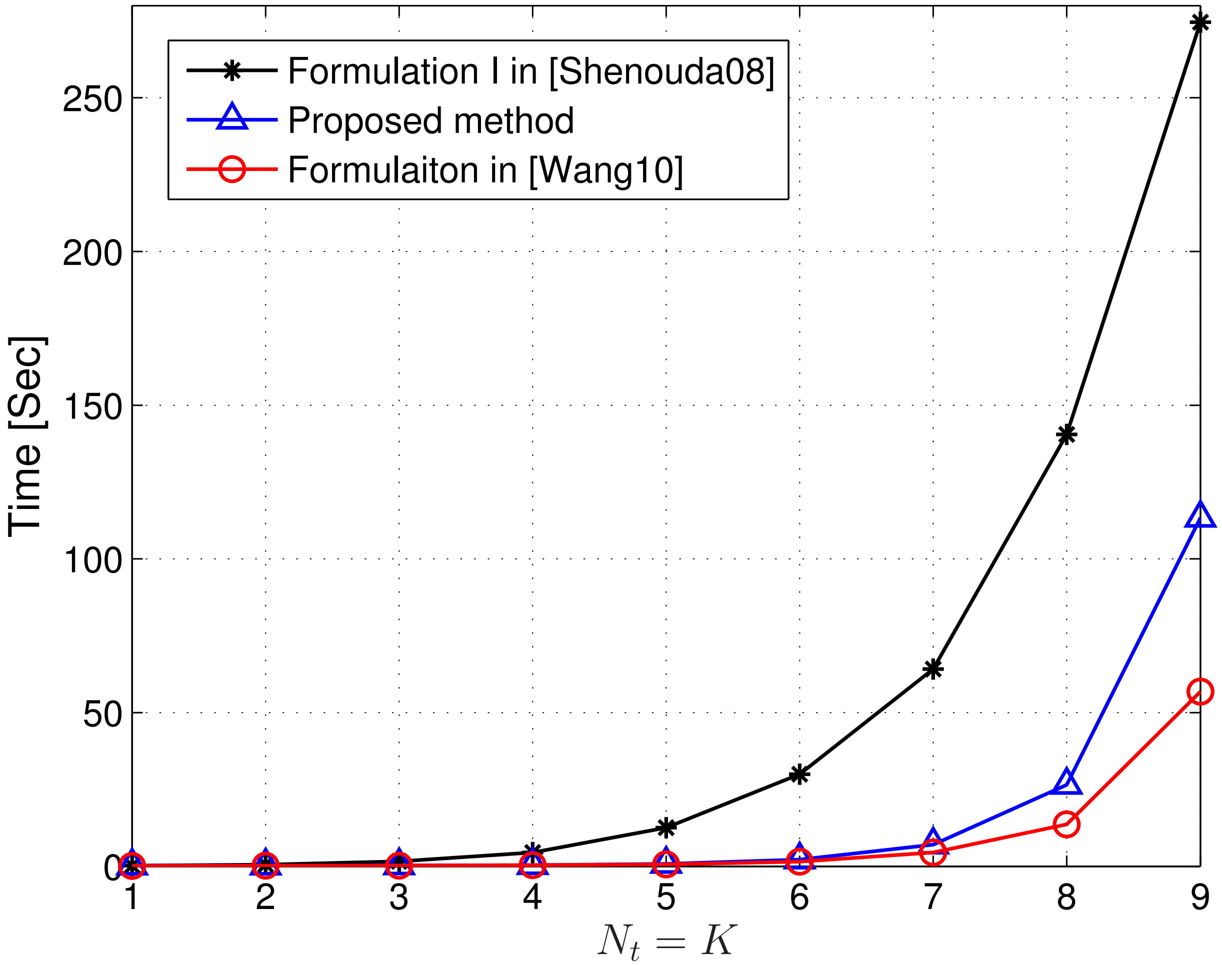
Average Transmission Power versus Target SINR



Computation Time Comparison($\gamma_i = 7$dB)



The average computation time of each method is obtained by averaging over 50 feasible channel realizations, on a laptop PC with 1.9GHz CPU and 4Gb RAM.

Conclusions

- A new approximation method for the probabilistic SINR constrained robust beamforming design problem has been proposed.
- The proposed method is based on
  - SDR
  - Bernstein-type inequality
- Simulation results show that the proposed method outperforms the existing methods in [Shenouda08] and [Wang10].

Thank you very much for your attention!

[Shenouda08] M. B. Shenouda and T. N. Davidson, Probabilistically-constrained approaches to the design of the multiple antenna downlink, in *Proc. 42nd Asilomar Conference*, Pacific Grove, Oct. 26-29, 2008, pp. 1120-1124.

[Wang10] K.-Y. Wang, T.-H. Chang, W.-K. Ma, and C.-Y. Chi, "A semidefinite relaxation based conservative approach to robust transmit beamforming with probabilistic SINR constraints," in *Proc. EUSIPCO*, Aalborg, Denmark, August 23-27, 2010, pp. 407-411.