Periodic CRB for Non-Bayesian Parameter Estimation

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Outline

Applications

Periodic parameter estimation arises in:

- Single/multiple tone estimation
- Direction-of-arrival (DOA) estimation
- Phase estimation

Applications

- Radar
- Sonar
- Communications
- Speech and audio signal processing

Example: Non-Bayesian phase estimation

The model:

$$x[n] = Ae^{i\theta} + w[n], \quad n = 0, \ldots, N-1$$

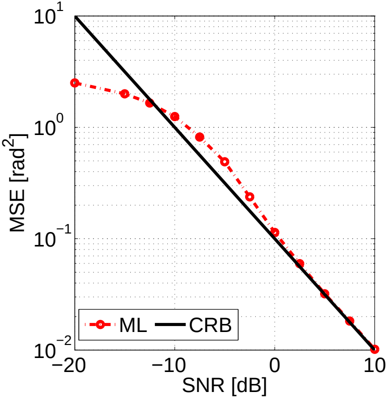
where

- $\theta \in [-\pi, \pi)$ - Unknown parameter
- A - Known amplitude
- $w[n] \sim \mathcal{CN}(0, \sigma^2)$ - i.i.d. noise
- $SNR = \frac{|A|^2}{\sigma^2}$
- Cramér-Rao bound:

$$CRB(\theta) = \frac{1}{2NSNR}$$

- Maximum-likelihood estimator:

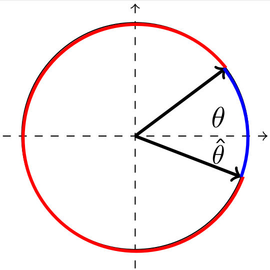
$$\hat{\theta}_{ML}(\mathbf{x}) = \angle\left(\frac{A^*}{N}\sum_{n=0}^{N-1} x[n]\right)$$



General periodic parameter estimation

The main problems

- The conventional MSE criterion is inappropriate.



- Error
- Modulo-2π error

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- The conventional MSE criterion is inappropriate.
- No uniformly unbiased estimator exists.

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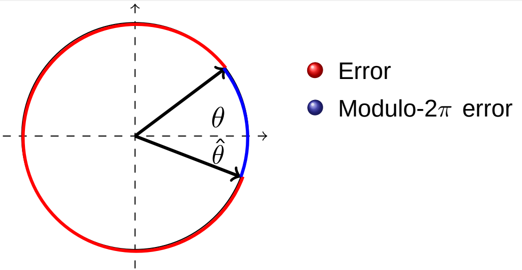

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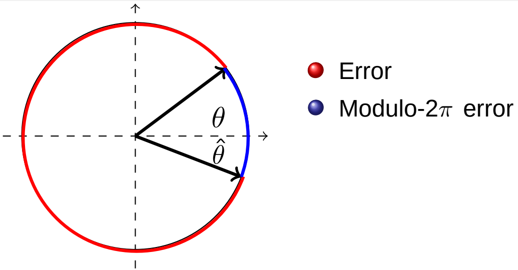
- The CRB may not be valid at low SNRs.



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In this work:

- Mean square periodic error (MSPE) criterion
- Periodic unbiasedness
- Periodic CRB

Non-Bayesian periodic parameter estimation

The model

- $\theta \in [-\pi, \pi)$ - Unknown deterministic parameter
- $(\Omega_{\mathbf{x}}, \mathcal{F}, P_\theta)$ - Probability space
- $\Omega_{\mathbf{x}}$ - Observation space
- $\{P_\theta\}$ - Family of probability measures parameterized by θ
- $\mathbf{x} \in \Omega_{\mathbf{x}}$ - Random observation vector
- $\hat{\theta}(\mathbf{x}) : \Omega_{\mathbf{x}} \to [-\pi, \pi)$ - estimator of θ

Note

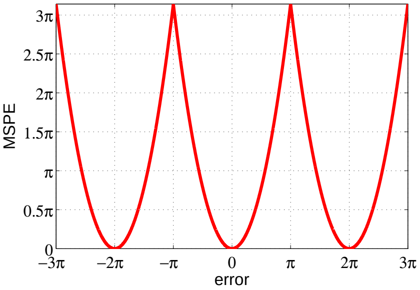
Even if the estimator is restricted to the region $[-\pi, \pi)$ the resulting error, $\hat{\theta}(\mathbf{x}) - \theta$, is in $[-2\pi, 2\pi)$!

The mean square periodic error (MSPE) criterion

The SPE cost function is

$$\mathrm{L}(\hat{\theta}(\mathbf{x}) - \theta) \triangleq \left(\text{mod}_{2\pi}\left[\hat{\theta}(\mathbf{x}) - \theta\right]\right)^2,$$

where the modulo-2π operator maps $\varepsilon \in \mathbb{R}$ to $[-\pi, \pi)$.



Lehmann unbiasedness

Definition

The estimator $\hat{\theta}(\mathbf{x})$ is Lehmann-unbiased w.r.t. the cost function $\mathrm{L}(\theta, \hat{\theta}(\mathbf{x}))$ if

$$\mathrm{E}_{\theta}\left[\mathrm{L}\left(\eta, \hat{\theta}(\mathbf{x})\right)\right] \geq \mathrm{E}_{\theta}\left[\mathrm{L}(\theta, \hat{\theta}(\mathbf{x}))\right], \quad \forall \eta, \theta \in \Theta$$

where $\mathrm{E}_{\theta}[\cdot]$ denotes expectation w.r.t. P_{θ} and Θ is the parameter space.

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Example: Mean-unbiasedness

Under the MSE cost function the Lehmann-unbiasedness is

$$\mathrm{E}_{\theta}\left[\hat{\theta}(\mathbf{x})\right] = \theta, \ \forall \theta \in \Theta .$$

Lehmann periodic-unbiasedness

The periodic-unbiasedness

Under the SPE cost function the Lehmann-unbiasedness is

$$\mathrm{E}_{\theta}\left[(\mathrm{mod}_{2\pi}[\hat{\theta}(\mathbf{x})-\eta])^2\right] \geq \mathrm{E}_{\theta}\left[(\mathrm{mod}_{2\pi}[\hat{\theta}(\mathbf{x})-\theta])^2\right], \ \forall \eta,\theta \in [-\pi,\pi).$$

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Under the SPE cost function the Lehmann-unbiasedness is

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For continuous estimators this periodic unbiasedness is:

$$1)\ \mathrm{E}_{\theta}[\text{mod}_{2\pi}[\hat{\theta}(\mathbf{x}) - \theta]] = 0, \quad \forall \theta \in [-\pi, \pi)$$
$$2)\ f_{\hat{\theta};\theta}\left(\text{mod}_{2\pi}\left[\theta + \pi\right]\right) < \frac{1}{2\pi}, \quad \forall \theta \in [-\pi, \pi)\ ,$$

where $f_{\hat{\theta};\theta}(\cdot)$ is the pdf of the estimator parameterized by θ.

Example: Non-Bayesian phase estimation

The model:

$$x[n] = Ae^{i\theta} + w[n], \quad n = 0, \ldots, N-1$$

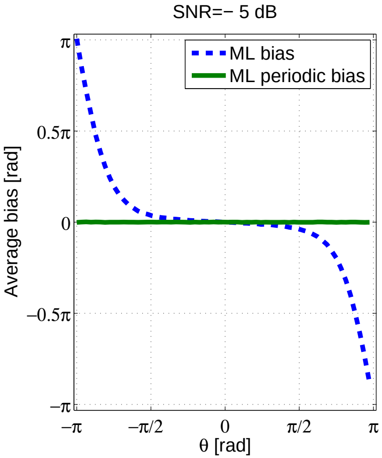
where $\theta \in [-\pi, \pi)$, A is known, and $w[n] \sim \mathcal{CN}\left(0, \sigma^2\right)$.

The estimators

$$\hat{\theta}_K(\mathbf{x}) = \angle\left(\frac{A^*}{K}\sum_{n=0}^{K-1} x[n]\right), \; K = 1, \ldots, N$$

are periodic-unbiased.

$\Rightarrow$ the ML estimator ($K = N$) is periodic-unbiased.



The periodic CRB

Theorem

Let $\hat{\theta}(\mathbf{x})$ be a periodic-unbiased estimator of $\theta \in [-\pi, \pi)$ and assume the "periodic-regularity conditions", then

$$\mathrm{E}_\theta \left[\left(\mathrm{mod}_{2\pi} \left[\hat{\theta}(\mathbf{x}) - \theta \right] \right)^2 \right] \geq PCRB(\theta)$$

where

$$PCRB(\theta) \triangleq CRB(\theta) \cdot \left(1 - 2\pi f_{\hat{\theta};\theta} \left(\mathrm{mod}_{2\pi} [\theta + \pi] \right) \right)^2$$

$$CRB(\theta) = \mathrm{E}_\theta^{-1} \left[\left(\frac{\partial \log f(\mathbf{x}; \theta)}{\partial \theta} \right)^2 \right] .$$

The periodic CRB (cont'd)

Properties

1. The PCRB is valid at any SNR.

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$$PCRB(\theta) \triangleq CRB(\theta) \cdot \left(1 - 2\pi f_{\hat{\theta};\theta}\left(\text{mod}_{2\pi}[\theta + \pi]\right)\right)^2$$

Due to the periodic-unbiasedness conditions: $< \frac{1}{2\pi}$

The periodic CRB (cont'd)

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Due to the periodic-unbiasedness conditions: $< \frac{1}{2\pi}$

However, the CRB is not a valid bound for periodic estimation!

The periodic CRB (cont'd)

Properties

3. The CRB for mean-biased estimators coincides with the PCRB for periodic-unbiased estimators.

MSE bound

biased-CRB

$+$

bias: $b(\theta) = 2\pi \times$
$(1 - F_{\hat{\theta};\theta}(\theta + \pi) - F_{\hat{\theta};\theta}(\theta - \pi))$

$\Leftrightarrow$

MSPE bound

PCRB

$+$

periodic bias = 0

* $F_{\hat{\theta};\theta}(\cdot)$ is the cdf of the estimator parameterized by θ.

The mixed PCRB bound

The model

- Parameter vector: $\boldsymbol{\theta} = [\theta_1, \theta_2]^T$, $\theta_1 \in [-\pi, \pi)$, $\theta_2 \in \mathbb{R}$
- Estimator: $\hat{\boldsymbol{\theta}}(\mathbf{x}) = [\hat{\theta}_1(\mathbf{x}), \hat{\theta}_2(\mathbf{x})]^T$, $\hat{\theta}_1(\mathbf{x}) \in [-\pi, \pi)$, $\hat{\theta}_2(\mathbf{x}) \in \mathbb{R}$
- Constraint: - Periodic-unbiasedness for θ_1
  - Mean-unbiasedness for θ_2

The mixed PCRB bound

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- Constraint: - Periodic-unbiasedness for θ_1
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The bound

$$\mathrm{E}_{\boldsymbol{\theta}}\left[\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T\right] \geq \mathbf{U}(\boldsymbol{\theta})\mathbf{J}^{-1}(\boldsymbol{\theta})\mathbf{U}(\boldsymbol{\theta})^T$$

- $\boldsymbol{\epsilon} = \left[\mathrm{mod}_{2\pi}\left[\hat{\theta}_1(\mathbf{x}) - \theta_1\right], \hat{\theta}_2(\mathbf{x}) - \theta_2\right]^T$
- $\mathbf{J}(\boldsymbol{\theta})$ - Fisher information matrix
- $\mathbf{U}(\boldsymbol{\theta}) = \mathit{diag}(1 - 2\pi f_{\hat{\theta}_1;\theta}(\mathrm{mod}_{2\pi}[\theta + \pi]), 1)$

Example

The model

$$\begin{aligned} x[n] &= Ae^{i(n\omega_0+\varphi)} + w[n], \\ n &= 0,\dots,N-1 \end{aligned}$$

where

- ω_0 - Known frequency
- $w[n] \sim \mathcal{CN}\left(0,\sigma^2\right)$ - i.i.d. noise
- $\boldsymbol{\theta} = [\varphi, A]^T$ - $\varphi \in [-\pi,\pi)$, $A \in \mathbb{R}$.
- Periodic-unbiased estimators:

$$\hat{\varphi}_K(\mathbf{x}) = \angle\left(\sum_{n=0}^{K-1} x[n]e^{-in\omega_0}\right),$$

$$K = 1,\dots,N$$

The PCRB

$$CRB(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\sigma^2}{2A^2N} & 0 \\ 0 & \frac{\sigma^2}{2N} \end{bmatrix}$$

$$PCRB(\boldsymbol{\theta}) = \begin{bmatrix} \frac{\sigma^2}{2NA^2}C_N^2 & 0 \\ 0 & \frac{\sigma^2}{2N} \end{bmatrix}$$

where

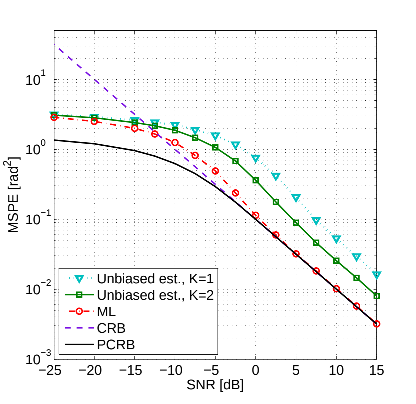
$$C_N \triangleq f_{\hat{\varphi}_N;\varphi}(\mathrm{mod}_{2\pi}[\varphi+\pi]) = \frac{1}{2\pi}\left(e^{-\frac{NA^2}{\sigma^2}} - \sqrt{\pi\frac{NA^2}{\sigma^2}}\,erfc\left(\sqrt{\frac{NA^2}{\sigma^2}}\right)\right).$$

Example

Results

- The bounds and the MSPE of the estimators do not depend on φ.
- CRB and PCRB are achievable by the ML estimator for high SNRs.
- The conventional CRB does not provide a valid lower bound for low SNRs.

- $N = 5$ samples
- $\omega_0 = 0.5$



Conclusion

- The concept of *non-Bayesian* periodic parameter estimation was introduced.
- The periodic-unbiasedness for the SPE cost function was derived using Lehmann unbiasedness definition.
- The periodic versions of the CRB for periodic parameter and mixed vector were developed.
- The PCRB provides a valid bound in periodic problems.
- Periodic-unbiased estimators and the PCRB for phase estimation problem were derived.

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In work:

- Barankin-like periodic bounds
- Hybrid bounds
- Periodic minimax estimation

The Regularity conditions

"Periodic regularity conditions":

1. $\frac{\partial \log f(\mathbf{x};\theta)}{\partial \theta}$ exists and is finite $\forall f(\mathbf{x};\theta) > 0, \quad \forall \theta \in [-\pi, \pi)$
2. $\mathrm{E}_\theta \left[\frac{\partial \log f(\mathbf{x};\theta)}{\partial \theta} \right] = 0, \quad \forall \theta \in [-\pi, \pi)$
3. $\int_{\Omega_\mathbf{x}} \frac{\partial}{\partial \theta} \mathrm{mod}_{2\pi} \left[\hat{\theta}(\mathbf{x}) - \theta \right] f(\mathbf{x};\theta)\, d\mathbf{x} = \frac{\partial}{\partial \theta} \mathrm{E} \left[\mathrm{mod}_{2\pi} \left[\hat{\theta}(\mathbf{x}) - \theta \right] \right],$
   for any periodic-unbiased estimator, $\hat{\theta}$.

The mean-bias of periodic-unbiased estimators

The periodic-unbiasedness 1'st condition:

$$\begin{aligned}
\mathrm{E}_\theta[\mathrm{mod}_{2\pi}[\hat{\theta}(\mathbf{x}) - \theta]] &= 0 \\
\Rightarrow b(\theta) \stackrel{\triangle}{=} \mathrm{E}_\theta[\hat{\theta}] - \theta &= 2\pi\left(1 - F_{\hat{\theta};\theta}(\theta+\pi) - F_{\hat{\theta};\theta}(\theta-\pi)\right) \\
\Rightarrow \frac{\mathrm{d}b(\theta)}{\mathrm{d}\theta} &= -2\pi f_{\hat{\theta};\theta}(\mathrm{mod}_{2\pi}[\theta+\pi])
\end{aligned}$$

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The CRB on the MSE of any *biased* estimator is given by

$$MSE(\hat{\theta}) \geq BCRB(\theta) = CRB(\theta) \cdot \left(1 + \frac{\mathrm{d}b(\theta)}{\mathrm{d}\theta}\right)^2$$

and by substituting $\frac{\mathrm{d}b(\theta)}{\mathrm{d}\theta}$ in $BCRB(\theta)$, one obtains

$$BCRB(\theta) = CRB(\theta) \cdot \left(1 - 2\pi f_{\hat{\theta};\theta}(\mathrm{mod}_{2\pi}[\theta + \pi])\right)^2 = PCRB(\theta).$$

Bounds improvements

1. Optimal bound: Let $\hat{\theta}_{opt}$ be the minimum MSPE periodic-unbiased estimator of θ:

$$B_2(\theta) = CRB(\theta) \cdot \left(1 - 2\pi f_{\hat{\theta}_{opt};\theta}\left(\mathrm{mod}_{2\pi}\left[\theta+\pi\right]\right)\right)^2$$

2. Asymptotic bound: Let $f_{\hat{\theta}_{ML,\infty};\theta}\left(\mathrm{mod}_{2\pi}\left[\theta+\pi\right]\right)$ be the asymptotic pdf of the ML estimator:

$$B_3(\theta) = CRB(\theta) \cdot \left(1 - 2\pi f_{\hat{\theta}_{ML,\infty};\theta}\left(\mathrm{mod}_{2\pi}\left[\theta+\pi\right]\right)\right)^2$$

3. Minimal bound: Let $\mathcal{T}$ be the set of all periodic-unbiased estimators of θ:

$$B_1(\theta) = \min_{\left\{\eta(\hat{\theta},\theta),\ \hat{\theta}\in\mathcal{T}\right\}} CRB(\theta) \cdot \left(1 - 2\pi\eta(\hat{\theta})\right)^2$$

where $\eta(\hat{\theta},\theta) = f_{\hat{\theta};\theta}\left(\mathrm{mod}_{2\pi}\left[\theta+\pi\right]\right)$.

Minimax

The minimax estimator is the estimator that minimizes the worst-case MSE among all possible values of θ in Θ. That is,

$$\hat{\theta}_{minimax} = \min_{\hat{\theta}} \max_{\theta \in \Theta} \mathrm{E}_{\theta}\left[\left(\hat{\theta} - \theta\right)^2\right]$$

In similar, we can define the periodic-minimax estimator:

$$\hat{\theta}_{P-minimax} \min_{\hat{\theta}} \max_{\theta \in [-\pi,\pi)} \mathrm{E}_{\theta}\left[\left(\mathrm{mod}_{2\pi}\left[\hat{\theta}(\mathbf{x}) - \theta\right]\right)^2\right]$$