Spasity-based Sinogram Denoising for Low-Dose Computed Tomography

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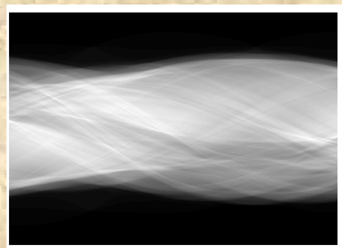
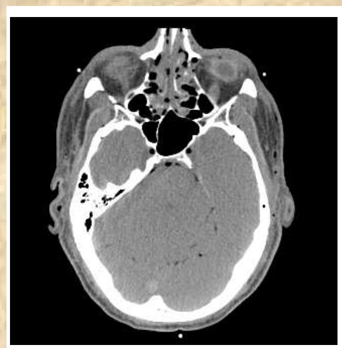




Computed Tomography – model.

Photon count y_l - modeled as an instance of the Poisson variable

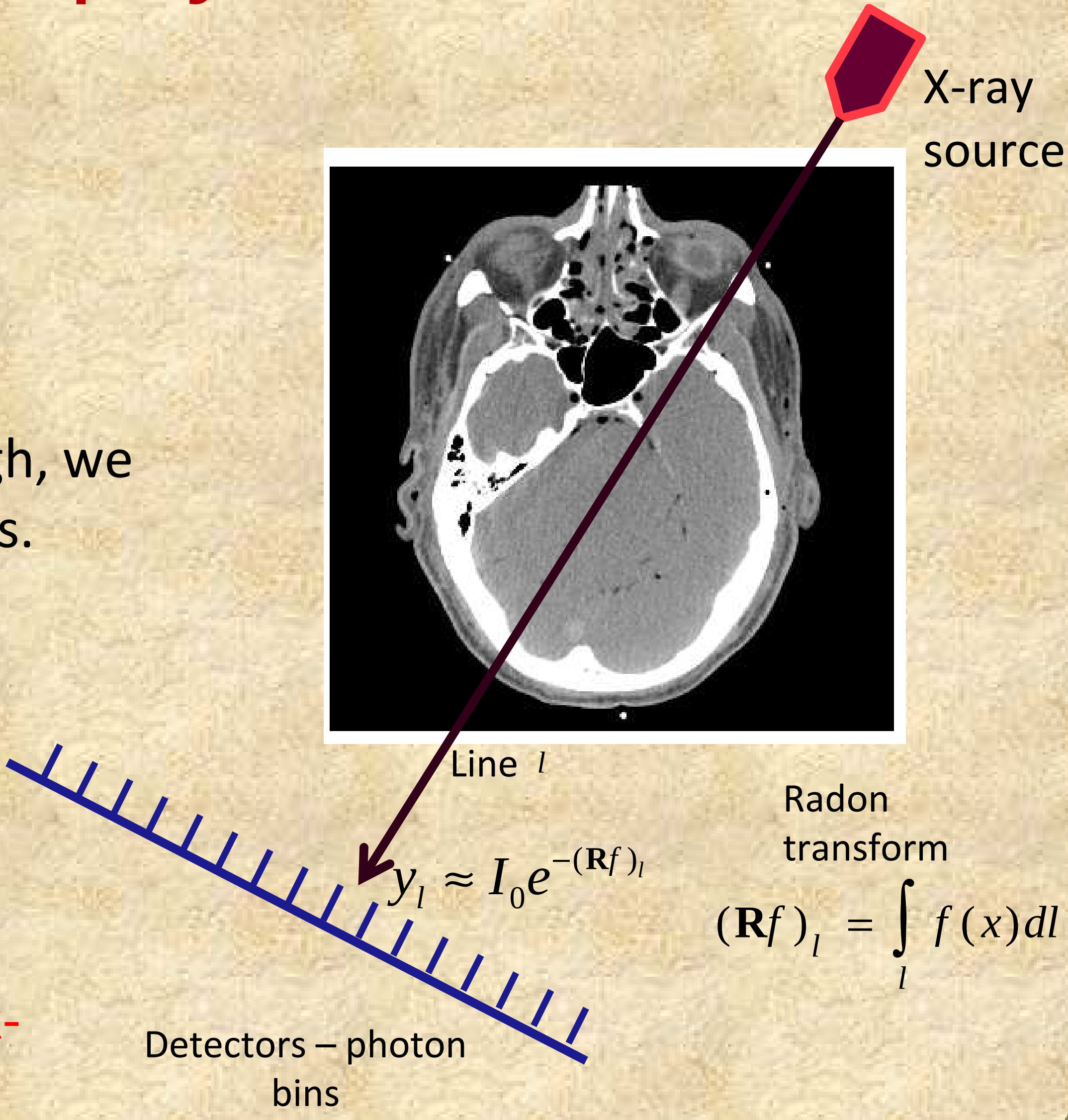
$$Y_l \sim Poisson(\lambda_l), \quad \lambda_l = I_0 e^{-(\mathbf{R}f)_l}$$

Exposure – quality tradeoff: if I_0 is high, we get great images and radiated patients.

Solution 1: statistically-based iterative algorithms.

Solution 2: radiate a local region.

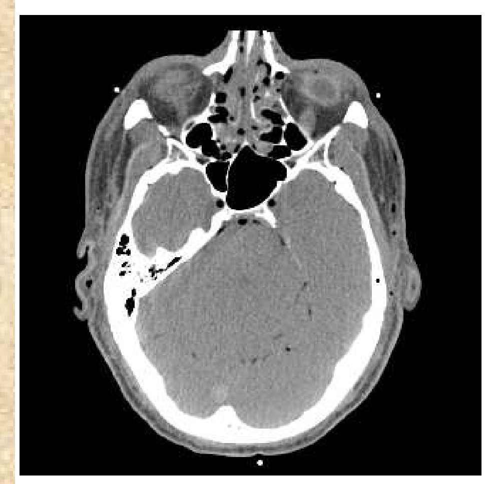
Solution 3: enhance the Filtered Back-Projection using data processing.



Radon transform

$$(\mathbf{R}f)_l = \int_l f(x)\,dl$$

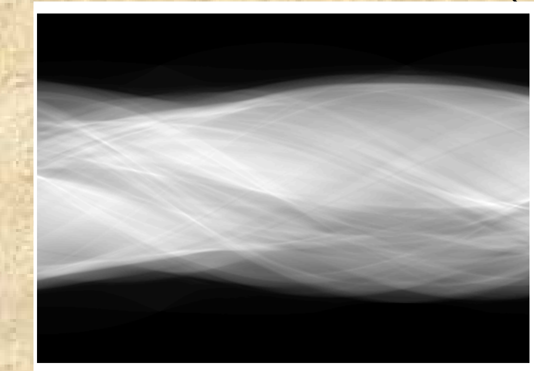
Computed Tomography – model.

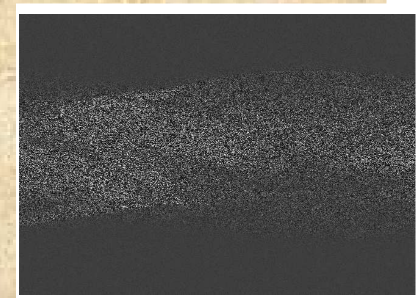


$$g_l = (Rf)_l = -\log\left(\frac{\lambda_l}{I_0}\right)$$

Ideal CT scan
+ log transform

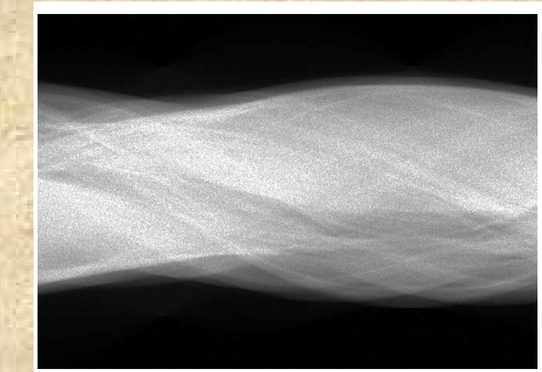
$f \rightarrow g$





+

Low-count
Poisson noise
(data dependent)



$\tilde{g}$

Reconstruction
algorithm

(default: Filtered
Back-Projection)

Sparse-Land model for signals

The concept: natural signals admit a faithful representation using only few columns (atoms) from a dedicated overcomplete dictionary.

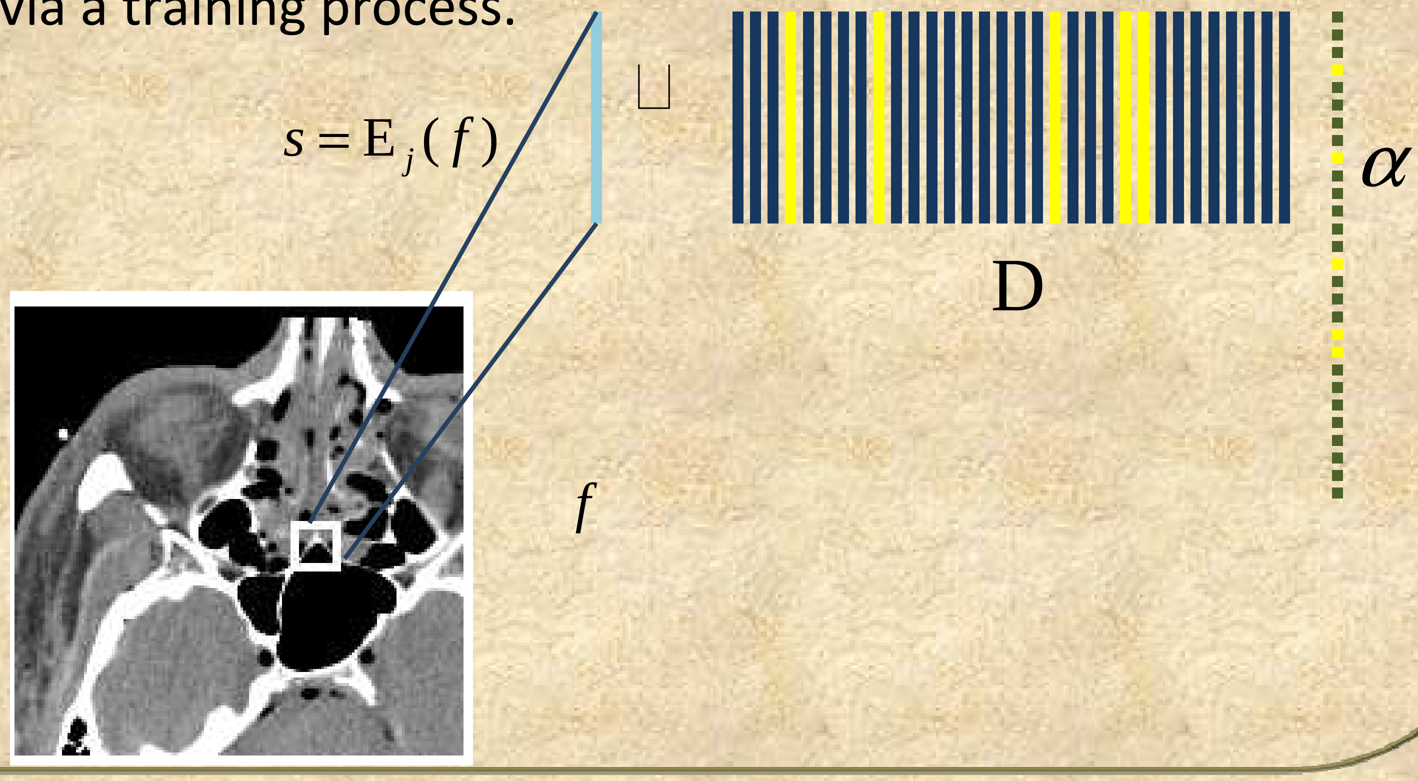
Natural dictionaries:
Wavelets, Discrete Cosines, Fourier.

Dictionaries tailored to the specific family of signals: obtained via a training process.

$\|\alpha\|_0 \leq k$ Number of non-zeros is small

$\|v\|_2 \leq \varepsilon$ Residual is small

$$s + v = \mathrm{D}\alpha$$



Sparse-Land paradigm

Denoising technique (Elad, Aharon, 2006):

$$\left\{\mathrm{D}^*, f^*, \{\alpha^*\}_j\right\} = \arg \min_{\mathrm{D}, f, \{\alpha\}_j} \left\{ \delta \left\| f - \tilde{f} \right\|_2^2 + \sum_j \mu_j \left\| \alpha_j \right\|_0 + \sum_j \left\| \mathrm{D}\alpha_j - \mathrm{E}_j(f) \right\|_2^2 \right\}$$

1. (K-SVD) Train a dictionary D along with sparse representations $\{\alpha\}_j$:

$$\left\{\mathrm{D}^*, \{\alpha^*\}_j\right\} = \arg \min_{\mathrm{D}, \{\alpha\}_j} \left\{ \sum_j \mu_j \left\| \alpha_j \right\|_0 + \sum_j \left\| \mathrm{D}\alpha_j - \mathrm{E}_j(f) \right\|_2^2 \right\}$$

Sparse coding: $\alpha_j = \arg \min_{\alpha} \left\| \alpha \right\|_0 \quad s.t. \left\| \mathrm{D}\alpha - \mathrm{E}_j(f) \right\|_2^2 \leq \varepsilon_j$

2. Compute the image estimate:

$$f^* = \arg \min_{f} \left\{ \delta \left\| f - \tilde{f} \right\|_2^2 + \sum_j \left\| \mathrm{D}\alpha_j - \mathrm{E}_j(f) \right\|_2^2 \right\}$$

Application to CT reconstruction

Previous work (Liao, Sapiro, 2007):

- Patch-wise sparse coding of CT image f.
- Online learning from noisy data.

$$\left\{ \mathrm{D}^*, f^*, \{\alpha^*\}_j \right\} = \arg\min_{\mathrm{D}, f, \alpha} \left\{ \delta \left\| \mathrm{R} f - g \right\|_2^2 + \sum_j \mu_j \left\| \alpha_j \right\|_0 + \sum_j \left\| \mathrm{D}\alpha_j - \mathrm{E}_j(f) \right\|_2^2 \right\}$$

- Fidelity term in the sinogram domain: $\left\| \mathrm{R} f - g \right\|_2^2$

➡

- Suboptimal, since Radon is ill-conditioned
- Assuming uniform noise

- Representations of patches in image domain: unknown and non-uniform sparsity weights μ_j

➡

- Sparsity assumption is mistreated

What can be done using FBP and supervised learning?

Proposed algorithm

• Sparse-coding in sinogram domain.	• Off-line training stage.
• Penalty in image domain.	• Use a set of high-quality images $\{ f \}$ and corresp. low-dose sinograms $\{ g \}$.

Off-line training: build dictionaries $\mathrm{D}_1, \mathrm{D}_2$:

Matrices W,Q are explained in the sequel.

Training stage I: penalty in sinogram domain

$$\left\{ \mathrm{D}_1, \left\{ \alpha_j^{g} \right\} \right\} = \arg\min_{\mathrm{D},\alpha_j^{g}} \left\{ \sum_{g} \left(\mu \sum_j \left\| \alpha_j^{g} \right\|_0 + \sum_j \left\| \mathrm{D}\,\alpha_j^{g} - \mathrm{E}_j(g) \right\|_{2,W}^2 \right) \right\}$$

Sparse coding: $\alpha_j^{g} = \arg\min_{\alpha} \left\| \alpha \right\|_0 \quad s.t. \left\| \mathrm{D}_1 \alpha - \mathrm{E}_j(g) \right\|_{2,W}^2 \le q$

Training stage II: penalty in image domain

q – Size of the patch

$$\mathrm{D}_2 = \arg\min_{\mathrm{D}} \left\{ \sum_{g} \left\| \mathrm{T}\, \mathrm{M}^{-1} \sum_j E_j^{\mathrm{T}} (\mathrm{D}\,\alpha_j^{g}) - f^{g} \right\|_{2,Q}^2 \right\}$$

T – FBP

$\mathrm{M} = \sum_j E_j^{\mathrm{T}} E_j$

recovered sinogram

Proposed algorithm

On-line reconstruction chain:

1. Use D_1 to compute patchwise sparse representations $\{\alpha_j\}$ of the noisy sinogram:

$$\alpha_j = \arg\min_{\alpha} \|\alpha\|_0 \quad s.t. \left\|\mathrm{D}_1 \alpha - \mathrm{E}_j(g)\right\|_{2,W}^2 \leq q$$

2. Use $\{\alpha_j\}$ and D_2 to compute the restored sinogram:

$$g_{rec} = \arg\min_{g} \sum_j \left\|\mathrm{D}_2 \alpha_j - \mathrm{E}_j(g)\right\|_2^2 = \mathrm{M}^{-1} \sum_j E_j^{\mathrm{T}} (\mathrm{D}_2 \alpha_j)$$

3. Apply standard reconstruction algorithm (FBP):

$$f_{rec} = \mathrm{T}(g_{rec})$$

Sparse-coding of sinogram data

$$\left\{ \mathrm{D}_1, \left\{ \alpha_j^{\tilde{g}} \right\} \right\} = \arg\min_{\mathrm{D}, \alpha_j^{\tilde{g}}} \left\{ \sum_{\tilde{g}} \left(\mu \sum_j \left\| \alpha_j^{\tilde{g}} \right\|_0 + \sum_j \left\| \mathrm{D}\, \alpha_j^{\tilde{g}} - \mathrm{E}_j(\tilde{g}) \right\|_{2,W}^2 \right) \right\}$$

Classical sparse coding: $$\alpha_j^{\tilde{g}} = \arg\min_{\alpha} \left\| \alpha \right\|_0 \quad s.t. \left\| \mathrm{D}\, \alpha - \mathrm{E}_j(\tilde{g}) \right\|_2^2 \leq \varepsilon$$

Statistical model of the noise implies

$$\mathrm{var}(g_l - \tilde{g}_l) = \lambda_l^{-1} \approx y_l^{-1}$$

True sinogram value

Measured sinogram value

True photon count

Measured photon count

Therefore $$\mathrm{E}\left(\sum_{l=1}^{q} \lambda_l (g_l - \tilde{g}_l)^2 \right) = q.$$ Define $$W = diag(y_1, y_2, \ldots, y_N)$$

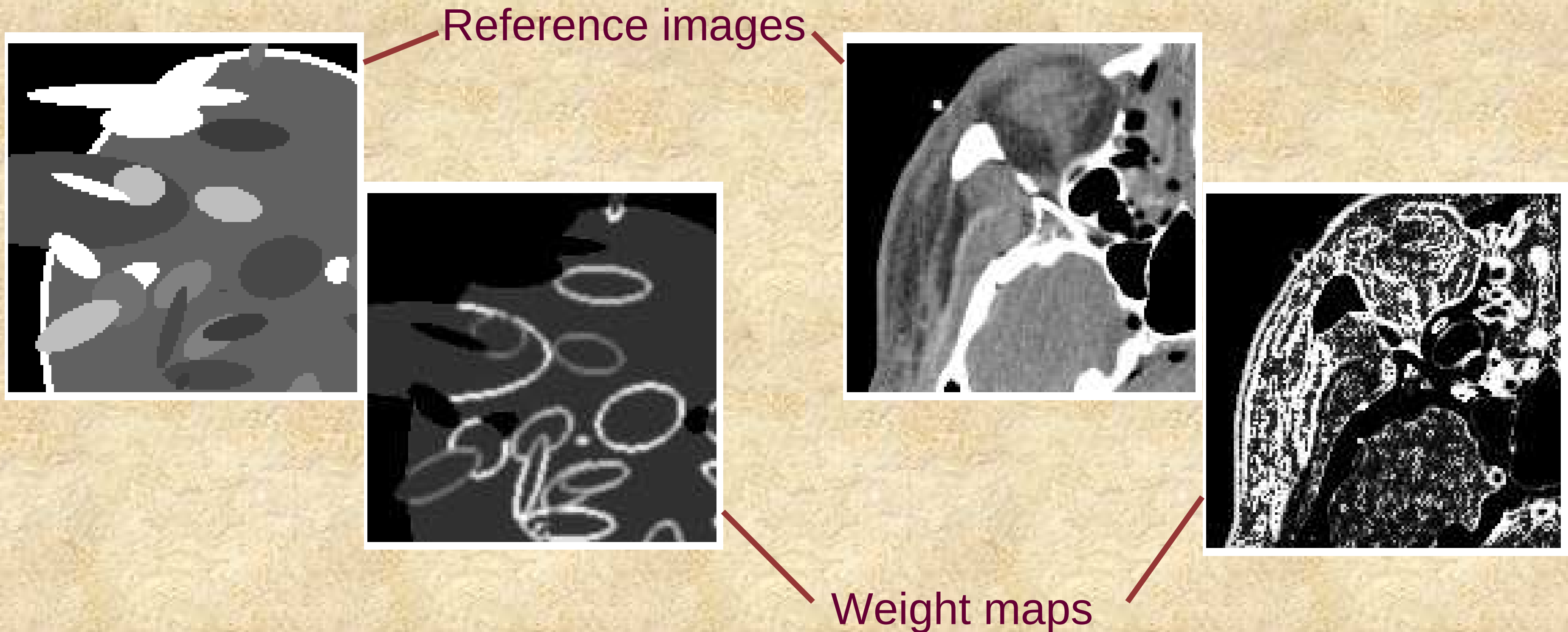
$$\alpha_j^{\tilde{g}} = \arg\min_{\alpha} \left\| \alpha_j \right\|_0 \quad s.t. \left\| \mathrm{D}\, \alpha_j^{\tilde{g}} - \mathrm{E}_j(\tilde{g}) \right\|_{2,W}^2 \leq q$$

q= # of pixels in a patch

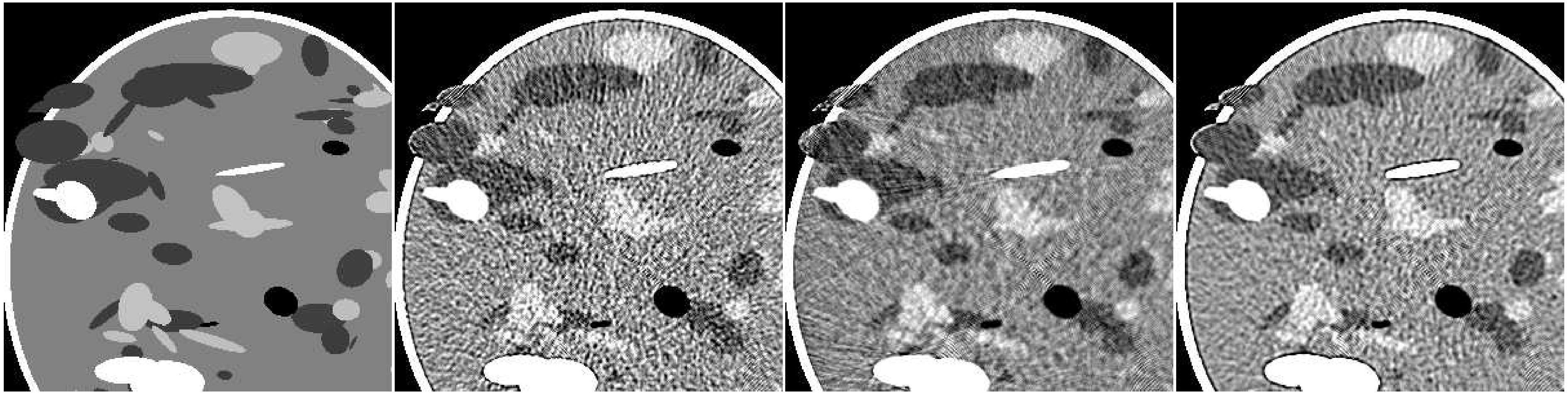
Error measure in image domain

Use a weighted L2 norm $\|x\|_{2,Q}^2$ for the difference between the reference image and the reconstructed one, in the training stage.

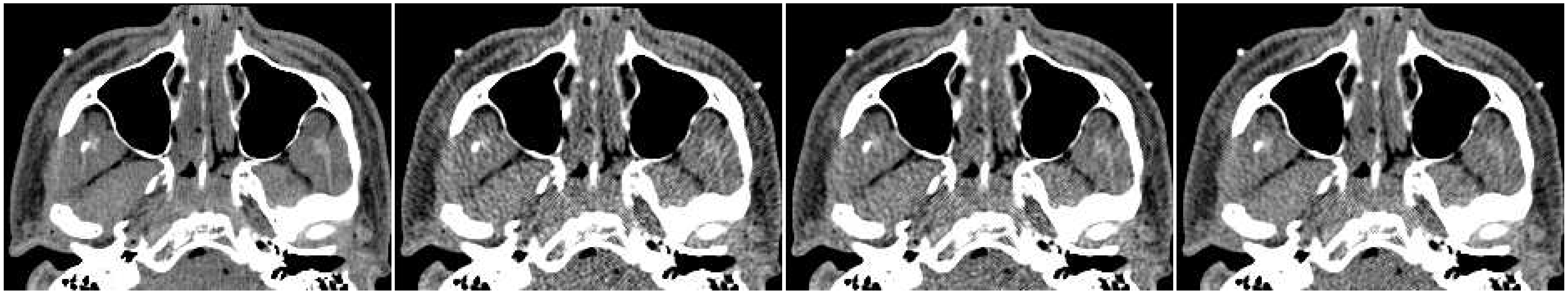
- Focus on regions with CT numbers corresponding to soft tisues.
- Enhance low-contrast edges.



Visual results



True image	FBP reconstruction	Our method	Double-dose FBP
	SNR=16.9 dB Q-norm = 2.88	SNR=17.6 dB Q-norm = 1.36	SNR=18.0 dB Q-norm = 1.98
	SNR=27.83 dB Q-norm =25.77	SNR=28.85 dB Q-norm = 19.34	SNR=28.42 dB Q-norm = 18.91



Summary

The performance of the fast, simple and popular FBP algorithm can be improved using signal-processing techniques based on supervised learning.

Sparse representations efficiently capture the sinogram data; tuning up the dictionaries for low error in image domain helps promoting imporant features. The algorithm shows promising results in domains of raw data and CT images.

The proposed algorithm is easily incorporated in existing CT scanners, and only require initial tune-up and training session. Then the images are produced almost at the same speed, with improved image quality.

Thank you.