Fourier Expansion of Hammerstein Models for Nonlinear Acoustic System Identification

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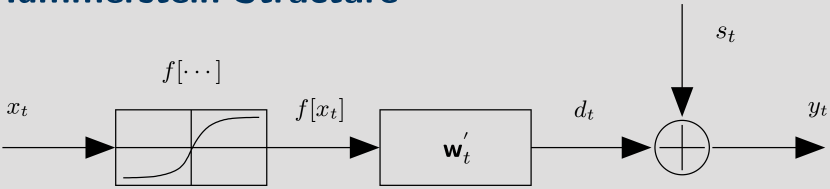
Outline

Identification of Nonlinear Acoustic Systems

- **A Hammerstein structure enables the inclusion of:**
  - Memoryless nonlinearity
  - Linear FIR system
- **Possibilities of modeling the nonlinearity in the system via, e.g.,:**
  - Power series, i.e., polynomial expansion basis (*traditional*)
  - Odd Fourier series, i.e., an orthogonal expansion basis (*proposed*)
- **Effects on ensuing equivalent multichannel system identification:**
  - Quality and rate of convergence
  - Learning of the underlying system nonlinearity

Hammerstein Structure



- x_t: Input signal
- $f[\cdots]$: Memoryless nonlinearity
- $f[x_t]$: Nonlinearly mapped input signal
- $\mathbf{w}_t^{'}$: Linear FIR system
- s_t: Observation noise
- y_t: Observation

Outline

Resolution of Nonlinearity in Hammerstein Model

- **Expansion model $f[x_t]$ of the input signal x_t:**

$$f[x_t] = \sum_{i=1}^{p} a_i \phi_i(x_t)$$

where $\phi_i(x_t)$ and p denotes the basis function and the order of expansion, respectively

- **Types of basis functions:**
  - Polynomial basis (*traditional*), i.e., $f[x_t] = \sum_{i=1}^{p} a_i x_t^i$

    where a_i is the i-th polynomial coefficient
  - Fourier basis, i.e., $f[x_t] = \sum_{i=1}^{p} a_i \cdot \sin(\pi \cdot i \cdot \frac{x_t}{L})$

    where a_i is the i-th Fourier coefficient and $2L$ is the fundamental period

Computation of Nonlinear Expansion Coefficients

- **For a given function $f[x_t]$ in the data range, $x_t \in [-1, 1]$:**
  - Computation of Power series coefficients in the least squares sense via

$$a_i = \arg\min_{\tilde{a}_i} \int_{-1}^{+1} \left[f[x_t] - \sum_{i=1}^{p} \tilde{a}_i x_t^i \right]^2 dx_t \ , \forall\ i$$

  - Closed-form computation of odd Fourier series coefficients via

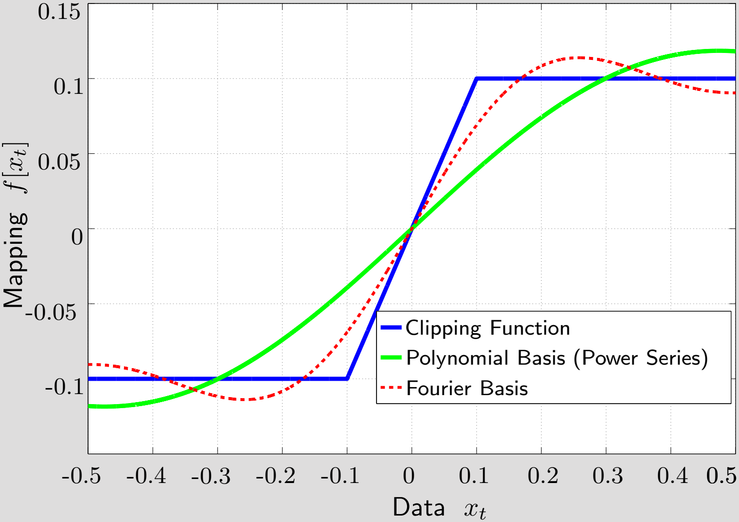
$$a_i = \frac{1}{L} \cdot \int_{-L}^{+L} f[x_t] \cdot \sin(\pi \cdot i \cdot \frac{x_t}{L}) dx_t \ , \forall\ i$$

- **Example of the nonlinearity as a clipping function:**

$$f[x_t] = \begin{cases} x_t & \text{if } |x_t| \leq x_{max} \\ x_{max} & \text{if } x_t > x_{max} \\ -x_{max} & \text{if } x_t < x_{max} \end{cases}$$

where x_{max} is the clipping threshold

Fitting Ability: $x_{max} = 0.1$, $p = 5$, $L = 1.5$



Outline

Block-Based Nonlinearly Mapped Input Signal

- **Block-based input and nonlinearly mapped signal:**

$$\mathbf{x}_\tau = [x_{\tau R-M+1},\ x_{\tau R-M+2},\ \cdots,\ x_{\tau R}]^H$$

$$\mathbf{f}_\tau = \{\ f[x_{\tau R-M+1}],\ f[x_{\tau R-M+2}],\ \cdots,\ f[x_{\tau R}]\ \}^H$$

with R is the block-shift and M the block-size, with τ as the block-index

- **Compact notation for the linear combination:**

$$\begin{aligned}\mathbf{f}_\tau &= \left\{\ \sum_{i=1}^{p} a_i \phi_i(x_{\tau R-M+1}),\ \cdots,\ \sum_{i=1}^{p} a_i \phi_i(x_{\tau R})\ \right\}^H \\ &= \sum_{i=1}^{p} a_i \left\{\phi_i(x_{\tau R-M+1}),\ \phi_i(x_{\tau R-M+2}),\ \cdots,\phi_i(x_{\tau R})\right\}^H \\ &= \sum_{i=1}^{p} a_i \mathbf{x}_{\tau,i}\end{aligned}$$

where $\mathbf{x}_{\tau,i} = \{\phi_i(x_{\tau R-M+1}),\ \phi_i(x_{\tau R-M+2}),\ \cdots,\ \phi_i(x_{\tau R})\}^H$

DFT-Domain Single Channel Signal Model

- **Applying diagonalization and DFT-Matrix $\mathbf{F}_M$ to $\mathbf{f}_\tau$:**

$$\widetilde{\mathbf{X}}_\tau = \operatorname{diag}\{\mathbf{F}_M \mathbf{f}_\tau\} = \sum_{i=1}^{p} a_i \operatorname{diag}\{\mathbf{F}_M \mathbf{x}_{\tau,i}\} = \sum_{i=1}^{p} a_i \mathbf{X}_{\tau,i}$$

where $\mathbf{X}_{\tau,i} = \operatorname{diag}\{\mathbf{F}_M \mathbf{x}_{\tau,i}\}$

- **Modeling $M - R$ non-zero coefficients of the echo path $\mathbf{w}_\tau = \mathbf{w}'_{t=\tau R}$:**

$$\mathbf{W}_\tau = \mathbf{F}_M \begin{bmatrix} \mathbf{w}_\tau^H & \mathbf{0}_R \end{bmatrix}^H$$

- **Overlap-save convolution model of the observation $\mathbf{Y}_\tau = \mathbf{F}_M \mathbf{Q} \mathbf{y}_\tau$:**

$$\begin{aligned} \mathbf{Y}_\tau &= \mathbf{F}_M \mathbf{Q} \mathbf{Q}^H \mathbf{F}_M^{-1} \widetilde{\mathbf{X}}_\tau \mathbf{W}_\tau + \mathbf{S}_\tau \\ &= \mathbf{G} \widetilde{\mathbf{X}}_\tau \mathbf{W}_\tau + \mathbf{S}_\tau = \widetilde{\mathbf{C}}_\tau \mathbf{W}_\tau + \mathbf{S}_\tau \end{aligned}$$

where $\widetilde{\mathbf{C}}_\tau = \mathbf{G} \widetilde{\mathbf{X}}_\tau$ and $\mathbf{G} = \mathbf{F}_M \mathbf{Q} \mathbf{Q}^H \mathbf{F}_M^{-1}$. Symbol $\mathbf{Q}^H = (\mathbf{0} \;\; \mathbf{I}_R)^H$ is an $R \times M$ projection matrix and $\mathbf{y}_\tau = \{y_{\tau R - R + 1}, \; y_{\tau R - R + 2}, \; \cdots, \; y_{\tau R}\}^H$

Outline

Equivalent Multichannel Structure

- **Substituting $\widetilde{\mathbf{X}}_\tau = \sum_{i=1}^{p} a_i \mathbf{X}_{\tau,i}$ for making the coefficients visible:**

$$\mathbf{Y}_\tau = \mathbf{G}\widetilde{\mathbf{X}}_\tau \mathbf{W}_\tau + \mathbf{S}_\tau = \mathbf{G} \sum_{i=1}^{p} a_i \mathbf{X}_{\tau,i} \mathbf{W}_\tau + \mathbf{S}_\tau$$

- **Expressing using matrix-vector formulation and absorbing nonlinear coefficients into the echo path:**

$$\mathbf{Y}_\tau = \mathbf{G}[\mathbf{X}_{\tau,1} \ \mathbf{X}_{\tau,2} \ \cdots \ \mathbf{X}_{\tau,p}] \begin{bmatrix} a_1 \mathbf{I}_M \\ a_2 \mathbf{I}_M \\ \cdots \\ a_p \mathbf{I}_M \end{bmatrix} \mathbf{W}_\tau + \mathbf{S}_\tau = \mathbf{G}\underline{\mathbf{X}}_\tau \begin{bmatrix} a_1 \mathbf{W}_\tau \\ a_2 \mathbf{W}_\tau \\ \cdots \\ a_p \mathbf{W}_\tau \end{bmatrix} + \mathbf{S}_\tau$$

$$= \underline{\mathbf{C}}_\tau \underline{\mathbf{W}}_\tau + \mathbf{S}_\tau \ .$$

where $\underline{\mathbf{C}}_\tau = [\mathbf{C}_{\tau,1} \ \mathbf{C}_{\tau,2} \ \cdots \ \mathbf{C}_{\tau,p}]$, while $\mathbf{C}_{\tau,i} = \mathbf{G}\mathbf{X}_{\tau,i}$.

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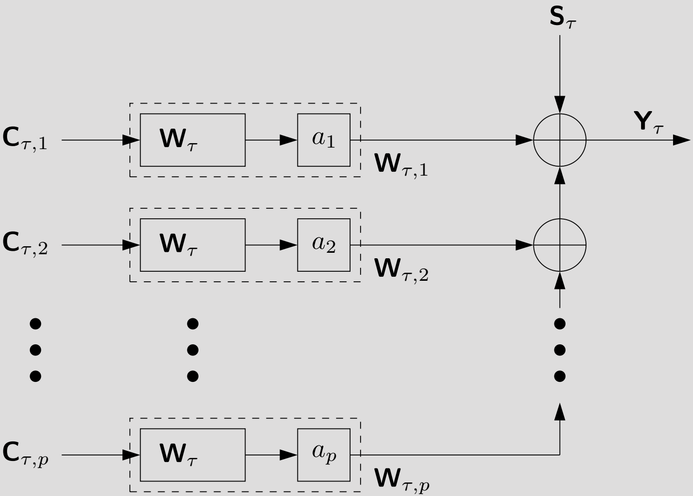
Equivalent Multichannel Signal Model



Outline

Multichannel Adaptive Algorithm

- **Multichannel frequency-domain adaptive filter (MCFDAF):**
  - Error and update equations of MCFDAF

$$\mathbf{E}_\tau = \mathbf{Y}_\tau - \underline{\mathbf{C}}_\tau \underline{\widehat{\mathbf{W}}}_{\tau-1}$$
$$\underline{\widehat{\mathbf{W}}}_\tau = \underline{\widehat{\mathbf{W}}}_{\tau-1} + \underline{\boldsymbol{\mu}}\underline{\mathbf{X}}_\tau^H \mathbf{E}_\tau$$

where stepsize for i-th channel $\boldsymbol{\mu}_{\tau,i}$ is given as a function of the power spectral density $\boldsymbol{\Psi}_{\underline{\mathbf{X}}_\tau \underline{\mathbf{X}}_\tau}$ and the individual adaptation constant in the range $0 < \alpha_i < 1$, i.e.,

$$\boldsymbol{\mu}_{\tau,i} = \alpha_i \boldsymbol{\Psi}^{-1}_{\underline{\mathbf{X}}_\tau \underline{\mathbf{X}}_\tau}$$

  - Computation of diagonal $M \times M$ matrix $\boldsymbol{\Psi}_{\underline{\mathbf{X}}_\tau \underline{\mathbf{X}}_\tau}$ by recursive averaging, i.e.,

$$\boldsymbol{\Psi}_{\underline{\mathbf{X}}_\tau \underline{\mathbf{X}}_\tau} = \gamma \boldsymbol{\Psi}_{\underline{\mathbf{X}}_{\tau-1} \underline{\mathbf{X}}_{\tau-1}} + (1-\gamma)\underline{\mathbf{X}}_\tau \underline{\mathbf{X}}_\tau^H$$

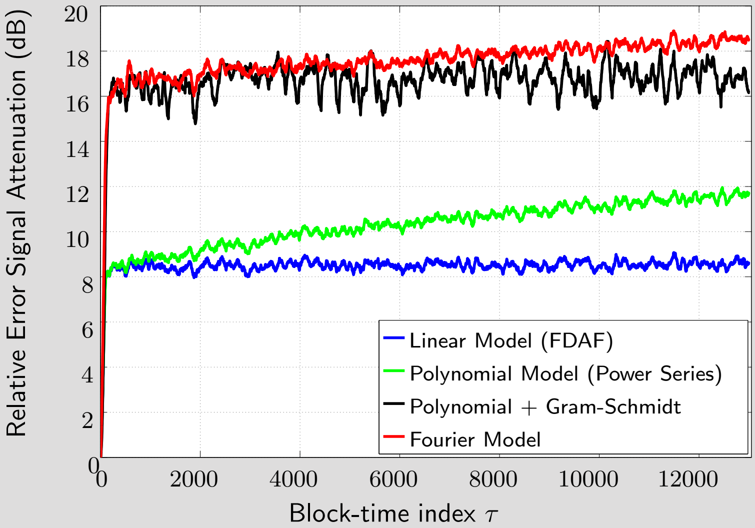
with γ as a forgetting factor in the range $0 < \gamma < 1$

Analysis Configurations

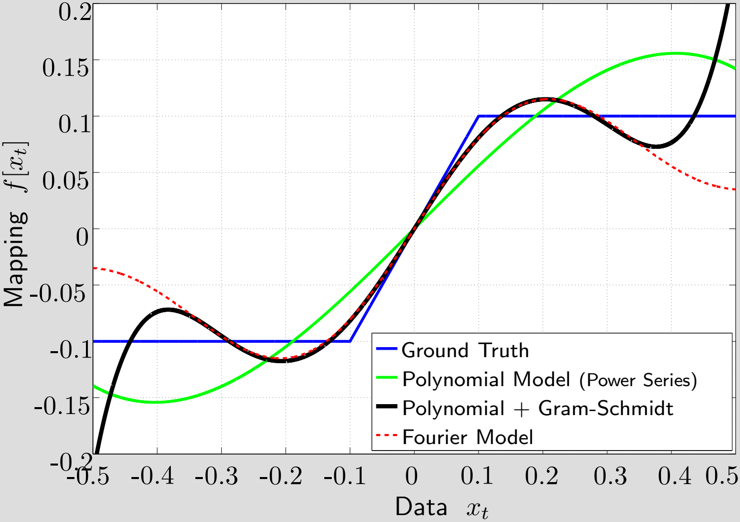
- **MCFDAF was configured to $M = 256$ and $R = 64$, respectively**
- **Linear-to-nonlinear power ratio SNR$_{NL}$ of $\sigma^2_{x_t}/\sigma^2_{x_t - f[x_t]} = 5$ and 20 dB**
- **Signal-to-observation noise power ratio set to SNR$= 60$ dB**
- **Two types of basis expansions:**
  - Power series
  - Odd Fourier series
- **Performance measured with respect to:**
  - Relative error signal attenuation ESA $= \sigma^2_{\mathbf{Y}_\tau}/\sigma^2_{\mathbf{E}_\tau}$
  - Inspection of the estimated nonlinear mapping $\widehat{f}[\cdots]$
- **Nonlinear coefficients extracted in least-squares sense:**
  - $a_i = (\mathbf{W}^H_\tau \widehat{\mathbf{W}}_{\tau,i})/(\mathbf{W}^H_\tau \mathbf{W}_\tau)$

    where $\widehat{\mathbf{W}}_{\tau,i}$ is the estimate of the i-th channel

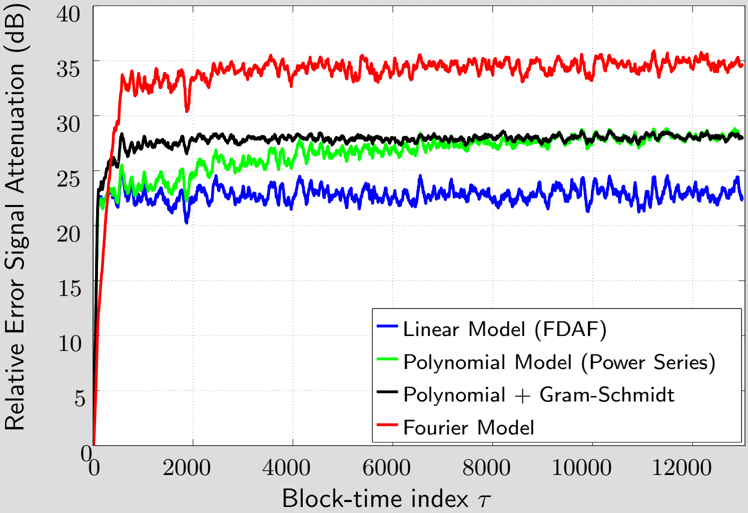
Performance Comparison: SNR$_{NL} = 5$ dB



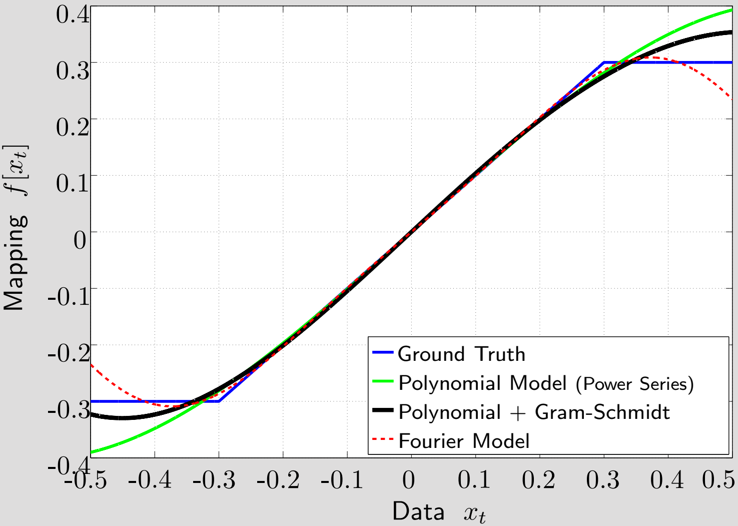
Extracted Nonlinear Mapping: $x_{max} = 0.1$



Performance Comparison: SNR$_{NL} = 20$ dB



Extracted Nonlinear Mapping: $x_{max} = 0.3$



Outline

Conclusions

- **Quasi-linear expansion of nonlinear Hammerstein models:**
  - Traditional Power series
  - Odd orthogonal Fourier series
- **Signal model in block frequency-domain:**
  - Contained basis-generic derivation
  - Efficient multichannel representation based on FFT
- **Results of multichannel adaptive identification:**
  - Orthogonal Fourier series lines up with polynomial modeling with Gramm-Schmidt orthogonalization
  - High error signal attenuation and effective imitation of the nonlinear response